Abstract. It has been shown previously by the author that any completely nonunitary \( C_{11} \) contraction with finite defect indices is reflexive. In this note we show that this is true even without the completely nonunitary assumption.

Recall that a bounded linear operator \( T \) on a complex, separable Hilbert space is reflexive if \( \text{Alg Lat} \ T = \text{Alg} \ T \), where \( \text{Alg Lat} \ T \) and \( \text{Alg} \ T \) denote, respectively, the weakly closed algebra of operators which leave invariant every invariant subspace of \( T \) and the weakly closed algebra generated by \( T \) and \( I \). It was shown in [9] that every completely nonunitary (c.n.u.) \( C_{11} \) contraction with finite defect indices is reflexive and it was conjectured that the same is true for arbitrary \( C_{11} \) contractions. In this note we move one step closer to establish this conjecture by dropping the completely nonunitary assumption, i.e. we prove that any \( C_{11} \) contraction with finite defect indices (a direct sum of a unitary operator and a c.n.u. \( C_{11} \) contraction) is reflexive. Note that this is not entirely trivial since in general we do not know whether the direct sum of two reflexive operators is reflexive (cf. [3, Question 2]).

In the discussion below we will follow the notations established in [9]. We also need some more facts from [10]. Let \( T \) be a c.n.u. \( C_{11} \) contraction with defect indices \( d_T = d_{1n} \equiv n < \infty \). Then \( T \) can be considered as defined on \( H \equiv [H_n^2 \oplus \Delta L_n^2] \oplus \{ \Theta_T w \oplus \Delta w : w \in H_n^2 \} \) by \( T(f \oplus g) = P(e^{it} f \oplus e^{it} g) \) for \( f \oplus g \in H \), where \( \Theta_T \) denotes the characteristic function of \( T \), \( \Delta = (I - \Theta_T^* \Theta_T)^{1/2} \) and \( P \) denotes the (orthogonal) projection onto \( H \). Since \( \Theta_T \) is outer from both sides, there exists an outer scalar multiple \( \delta \) of \( \Theta_T \) (cf. [7, p. 217]). Let \( \Omega \) be a contractive analytic function such that \( \Omega \Theta_T = \Theta_T \Omega = \delta I \). Let \( U \) denote the operator of multiplication by \( e^{it} \) on \( \Delta_{\ast} L_n^2 \), where \( \Delta_{\ast} = (I - \Theta_T \Theta_T^*)^{1/2} \), and let \( X : H \to \Delta_{\ast} L_n^2 \), \( Y : \Delta_{\ast} L_n^2 \to H \) be the operators defined by \( X(f \oplus g) = -\Delta f + \Theta_T g \) for \( f \oplus g \in H \) and \( Yu = P(0 \oplus \Omega u) \) for \( u \in \Delta_{\ast} L_n^2 \). Then \( X \) and \( Y \) are quasi-affinities which intertwine \( T \) and \( U \) and satisfy \( XY = \delta(T) \) and \( XY = \delta(U) \) (cf. [10, Lemma 2.1]).

Any absolutely continuous unitary operator \( U_\alpha \) on \( K \) is, by the spectral theorem, unitarily equivalent to the operator of multiplication by \( e^{it} \) on \( L^2(E_1) \oplus \cdots \oplus L^2(E_k) \), where \( k \) may be infinite and \( E_1, \ldots, E_k \) are Borel subsets of the
unit circle $C$ with $E_1 \supset E_2 \supset \cdots \supset E_k$. In particular, $U$ is unitarily equivalent to the operator of multiplication by $e^u$ on $L^2(F_1) \oplus \cdots \oplus L^2(F_k)$, where $C \supset F_1 \supset F_2 \supset \cdots \supset F_n$. Let $Z_1 : K \to L^2(F_1) \oplus \cdots \oplus L^2(E_k)$ and $Z_2 : \Delta E_n \to L^2(F_1) \oplus \cdots \oplus L^2(F_n)$ be the implementing unitary transformations.

Now we are ready to start. In the following lemmas we consider a $C_{11}$ contraction with finite defect indices whose unitary part is absolutely continuous. We first find operators in its double commutant. Lemma 2 deals with the reflexivity and the double commutant property.

**Lemma 1.** Let $S = U_a \oplus T$, where $U_a$ is an absolutely continuous unitary operator on $K$ and $T$ is a c.n.u. $C_{11}$ contraction with finite defect indices on $H$. Then \( \{S\}'' = \{\psi(U_a) \oplus P[\phi, \psi]\} : \psi \in L^\infty, A \Theta_T = \Theta_T A_0 \) and $B \Theta_T + \psi \Delta = \Delta A_0$ for some bounded analytic function $A_0$.

**Proof.** For any $V \in \{S\}''$, $V = V_1 \oplus V_2$ where $V_1 \in \{U_a\}''$ and $V_2 \in \{T\}''$. Hence

\[
V_1 = \psi_1(U_a) \quad \text{and} \quad V_2 = \begin{bmatrix} A & 0 \\ B & \psi_2 \end{bmatrix},
\]

where $\psi_1, \psi_2 \in L^\infty$ and $A, B$ satisfy $A \Theta_T = \Theta_T A_0$ and $B \Theta_T + \psi_2 \Delta = \Delta A_0$ for some bounded analytic function $A_0$ (cf. [9, Lemma 2]). Let $W = \delta(U_a)V_1 \oplus XV_2 Y \equiv W_1 \oplus W_2$. For any $u \in \Delta E_n$, we have

\[
W_2 u = XV_2 Y u = XP \begin{bmatrix} A & 0 \\ B & \psi_2 \end{bmatrix} \begin{bmatrix} 0 \\ \psi_2 u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \psi_2 u \end{bmatrix} = \Delta_0 u + \Theta_T \psi_2 \Omega u = \delta \psi_2 u.
\]

This shows that $W_2 = (\delta \psi_2)(U)$. Hence $W = (\delta \psi_1)(U_a) \oplus (\delta \psi_2)(U)$. Next we show that $W \in \{U_a \oplus U\}''$. Since $V_1 \in \{U_a\}''$ and $V_2 \in \{T\}''$, we have only to check that (i) any operator $Q : K \to L^2$ intertwining $U_a$ and $U$ intertwines $W_1$ and $W_2$ and (ii) any operator $R : H \to K$ intertwining $U$ and $U_a$ intertwines $W_2$ and $W_1$.

To prove (i), note that $YQ : K \to H$ intertwines $U_a$ and $T$. Since $V = V_1 \oplus V_2$ in $\{S\}''$, we have $YQV_1 = V_2 YQ$. Applying $X$ from the left on both sides, we obtain $XYQV_1 = XV_2 YQ$ or $\delta(U)QV_1 = W_2 Q$. But $\delta(U)QV_1 = Q(\delta(U_a)V_1 = QW_1$. Hence $Q$ intertwines $W_1$ and $W_2$, proving (i). (ii) can be proved in a similar fashion. Thus $W \in \{U_a \oplus U\}''$ as asserted and therefore $W = \xi(U_a \oplus U)$ for some $\xi \in L^\infty$. But we already have $W = (\delta \psi_1)(U_a) \oplus (\delta \psi_2)(U)$. It follows that $\xi = \delta \psi_1$ a.e. on $E_1$ and $\xi = \delta \psi_2$ a.e. on $F_1$, whence $\psi_1 = \psi_2$ a.e. on $E_1 \cap F_1$. Let $\psi$ in $L^\infty$ be such that $\psi = \psi_1$ a.e. on $E_1$ and $\psi = \psi_2$ a.e. on $F_1$. Then $V = \psi(U_a) \oplus P[\phi, \psi]$ as asserted.

For the converse, let $V = V_1 \oplus V_2 = \psi(U_a) \oplus P[\phi, \psi]$ for some $\psi \in L^\infty$. Again, we consider $W = \delta(U_a)V_1 \oplus XV_2 Y$. As before, it can be shown that $W = (\delta \psi)(U_a \oplus U) \in \{U_a \oplus U\}''$. Since $V_1 \in \{U_a\}''$ and $V_2 \in \{T\}''$ (cf. [9, Lemma 2]), to show that $V \in \{S\}''$ we have to check (i) any operator $Q : K \to H$ intertwining $U_a$ and $T$ intertwines $V_1$ and $V_2$ and (ii) any operator $R : H \to K$...
intertwining $T$ and $U_a$ intertwines $V_2$ and $V_1$. Here we only prove (i). Since $XQ: K \to \Delta_nL^2_n$ intertwines $U_a$ and $U$ and $W \in \{U_a \oplus U\}$, we have $XQ\delta(U_a)V_1 = XV_2YXQ$. It follows from the injectivity of $X$ that $Q\delta(U_a)V_1 = V_2YQ$. But $V_2YXQ = V_2\delta(T)Q = V_2Q\delta(U_a)$ and hence we have $QV_1\delta(U_a) = V_2Q\delta(U_a)$. Since $\delta(U_a)$ has dense range, we conclude that $QV_1 = V_2Q$ as asserted. Similarly for (ii). Hence $V \in \{S\}^\nu$, completing the proof.

**Lemma 2.** Let $S = U_a \oplus T$ be as in Lemma 1.

1. If $E_1 \cup F_1 \neq C$ a.e., then $\text{Alg Lat} S = \text{Alg} S = \{S\}^\nu$.
2. If $E_1 \cup F_1 = C$ a.e., then $\text{Alg Lat} S = \text{Alg} S = \{\varphi(S): \varphi \in H^\infty\}$.

In particular, $S$ is reflexive and $\{S\}^\nu = \text{Alg} S$ if and only if $E_1 \cup F_1 \neq C$ a.e.

**Proof.** (1) In this case, it suffices to show that $\text{Alg Lat} S \subseteq \{S\}^\nu$ and $\{S\}^\nu \subseteq \text{Alg} S$. To prove the former, let $V \in \text{Alg Lat} S$. Then $V = V_1 \oplus V_2$, where $V_1 \in \text{Alg Lat} U_a = \text{Alg} U_a$ and $V_2 \in \text{Alg Lat} T = \text{Alg} T$ since $U_a$ and $T$ are both reflexive (cf. [6] and [9]). Hence

$$V_1 = \psi_1(U_a) \quad \text{and} \quad V_2 = \begin{bmatrix} A & 0 \\ B & \psi_2 \end{bmatrix},$$

where $\psi_1, \psi_2 \in L^\infty$ and $A, B$ satisfy $A\Theta_T = \Theta_T A_0$ and $B\Theta_T + \psi_2\Delta = \Delta A_0$ for some $A_0$.

Consider the subspace

$$\mathcal{R} = \{Z_1^{-1}(\chi_{E_1}f \oplus \cdots \oplus \chi_{E_n}f) \oplus Z_2^{-1}(\chi_{F_1}f \oplus \cdots \oplus \chi_{F_n}f): f \in L^2\}$$

of $K \oplus \Delta_nL^2_n$. Note that $\mathcal{R}$ is a (closed) invariant subspace for $U_a \oplus U$. Let $\mathcal{R} = (\delta(U_a) \oplus Y)\mathcal{R}$. Then $\mathcal{R}$ is invariant for $S$ and hence $\overline{V\mathcal{R}} \subseteq \mathcal{R}$. Applying $I \oplus X$ on both sides, we obtain $(I \oplus X)V\mathcal{R} \subseteq (I \oplus X)\mathcal{R}$. But

$$(I \oplus X)V\mathcal{R} = (I \oplus X)(V_1 \oplus V_2)(\delta(U_a) \oplus Y)\mathcal{R} = (\psi_1(U_a) \oplus \psi_2(U))\delta(U_a \oplus U)\mathcal{R},$$

where the last equality was proved in Lemma 1, and

$$(I \oplus X)\mathcal{R} = (I \oplus X)(\delta(U_a) \oplus Y)\mathcal{R} = \delta(U_a \oplus U)\mathcal{R}.$$ 

Since $\delta$ is an outer function, $\delta(U_a \oplus U)|\mathcal{R}$ is a quasi-affinity on $\mathcal{R}$ (cf. [10, Lemma 2.3]). We conclude from above that $(\psi_1(U_a) \oplus \psi_2(U))\mathcal{R} \subseteq \mathcal{R}$. Hence for any $f \in L^2$, there exists $\psi \in L^2$ such that $\chi_{E_1}\psi f = \chi_{E_1}\psi$ a.e. and $\chi_{F_1}\psi f = \chi_{F_1}\psi$ a.e. In particular, for $f \equiv 1$ this implies that $\psi_1 = \psi$ a.e. on $E_1$ and $\psi_2 = \psi$ a.e. on $F_1$. Therefore, $V = \psi(U_a) \oplus P[\chi_{E_1}\psi]\in \{S\}^\nu$ by Lemma 1.

Next we show that $\{S\}^\nu \subseteq \text{Alg} S$. Let $V \in \{S\}^\nu$. By [5, Theorem 7.1], it suffices to show that $\text{Lat} S^{(\alpha)} \subseteq \text{Lat} V^{(\alpha)}$ for any $n > 1$, where

$$S^{(\alpha)} = \bigoplus_{n=1}^\infty S \quad \text{and} \quad V^{(\alpha)} = \bigoplus_{n=1}^\infty V.$$ 

Since $S^{(\alpha)}$ is an operator of the same type as $S$ and $V^{(\alpha)} \in \{S^{(\alpha)}\}^\nu$, it is clear that we have only to check for $n = 1$, i.e. $\text{Lat} S \subseteq \text{Lat} V$. To prove this, let $\mathcal{R} \in \text{Lat} S$. By Lemma 1, $V = V_1 \oplus V_2 = (\psi(U_a) \oplus P[\chi_{E_1}\psi])$ for some $\psi \in L^\infty$ and $A, B$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Let $W = \delta(U_a)V_1 \otimes XV_2 Y$. As proved in Lemma 1, $W = (\delta(\psi)(U_a \otimes U))''$. Since by our assumption $E_1 \cup F_1 \neq C$ a.e., every invariant subspace for $U_a \otimes U$ is bi-invariant, i.e. invariant for any operator in $\{U_a \otimes U\}$". In particular, $\mathcal{H} \equiv (T \otimes X)\mathcal{H}$ is invariant for $W$, i.e. $W\mathcal{H} \subseteq \mathcal{H}$. Applying $\delta(U_a) \otimes Y$ on both sides, we obtain $(\delta(U_a) \otimes Y)W \mathcal{H} \subseteq (\delta(U_a) \otimes Y)\mathcal{H}$. But

$$
(\delta(U_a) \otimes Y)W \mathcal{H} = (\delta(U_a)V_1 \delta(U_a) \otimes YXV_2 YX)\mathcal{H}
$$

where the last equality follows from the fact that $\delta(U_a \otimes T)|\mathcal{H}$ is a quasi-affinity on $\mathcal{H}$. (This can be proved in the same fashion as [10, Lemma 2.3].) On the other hand, $(\delta(U_a) \otimes Y)\mathcal{H} = \delta(U_a \otimes T)\mathcal{H} = \mathcal{H}$. We conclude that $(V_1 \oplus V_2)\mathcal{H} \subseteq \mathcal{H}$ hence $\mathcal{H} \in \text{Lat} V$. This completes the proof of (1).

(2) As in (1), let

$$
V = \psi_1(U_a) \oplus P
$$

be an operator in Alg Lat $S$. This time we consider the subspace

$$
\mathcal{H} = \left\{ Z \left( \chi_{E_1} f \oplus \cdots \oplus \chi_{E_n} f \right) \oplus Z_2^{-1}(\chi_{F_1} f \oplus \cdots \oplus \chi_{F_n} f) : f \in H^2 \right\}
$$

of $K \oplus \Delta_2 L_2^2$. Since $E_1 \cup F_1 = C$ a.e., it is easy to check that $\mathcal{H}$ is closed and invariant for $U_a \otimes U$. As in the first part of (1), we derive that for any $f \in H^2$ there exists $\varphi \in H^2$ such that $\chi_{E_1} \psi_1 f = \chi_{E_1} \varphi$ a.e. and $\chi_{F_1} \psi_2 f = \chi_{F_1} \varphi$ a.e. Hence for $f \equiv 1$, we have $\psi_1 = \varphi$ a.e. on $E_1$ and $\psi_2 = \varphi$ a.e. on $F_1$. Therefore $V = \varphi(U_a) \oplus P[\varphi, \psi_2]$. Using the fact that $\{P(0 \oplus g) : g \in \Delta L_2^n\}$ is dense in $H$ (cf. [9, proof of Lemma 2]), we can easily show that $P[\varphi, \psi_2] = \varphi(T)$. Hence $V = \varphi(U_a \oplus T) = \varphi(S)$, completing the proof.

Now comes our main result.

**Theorem 3.** Any $C_{11}$ contraction $S$ with finite defect indices is reflexive. Moreover, $\{S\}'' = \text{Alg} S$ if and only if $E_1 \cup F_1 \neq C$ a.e.

**Proof.** Let $S = U_a \oplus U_a \oplus T$ on $L \oplus K \oplus H$ be such that $U_a$ and $U_a$ are singular and absolutely continuous unitary operators, respectively, and $T$ is a c.n.u. $C_{11}$ contraction (cf. [7, p. 9] and [4]). We first show that $\text{Alg} U_a \oplus \text{Alg}(U_a \oplus T) = \text{Alg} S$. By [5, Theorem 7.1], this is equivalent to $\text{Lat} U_a^{(n)} \oplus \text{Lat}(U_a \oplus T)^{(n)} = \text{Lat} S^{(n)}$ for all $n > 1$. Since $S^{(n)} = U_a^{(n)} \oplus (U_a \oplus T)^{(n)}$ is of the same type as $S = U_a \oplus (U_a \oplus T)$, it suffices to check for $n = 1$, i.e. $\text{Lat} U_a \oplus \text{Lat}(U_a \oplus T) = \text{Lat} S$. Let $\mathcal{H} \in \text{Lat} S$. We can decompose the $C_{11}$ contraction $S|\mathcal{H}$ as $S|\mathcal{H} = S_1 \oplus S_2 \oplus S_3$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where $S_1$ and $S_2$ are singular and absolutely continuous unitary operators and $S_3$ is a c.n.u. $C_1$ contraction. Note that $\mathcal{H}_1$ and $\mathcal{H}_2 \oplus \mathcal{H}_3$ are invariant for $S$. To complete the proof, we have to show that $\mathcal{H}_1 \subseteq L$ and $\mathcal{H}_2 \oplus \mathcal{H}_3 \subseteq K \oplus H$.

Let $W$ be the operator of multiplication by $e^{i\theta}$ on $L_2^2 \oplus \Delta L_2^2$. Then $Z \equiv U_a \oplus U_a \oplus W$ is the minimal unitary dilation of $S$. It follows that $Z$ is a unitary dilation of
$S_2 \oplus S_3$. There exists a reducing subspace $\mathcal{E}$ for $Z$ such that $Z|\mathcal{E}$ is the minimal unitary dilation of $S_2 \oplus S_3$ (cf. [7, p. 13]). Since $S_2$ is absolutely continuous and $S_3$ is c.n.u., $Z|\mathcal{E}$ must be absolutely continuous and continuous (cf. [7, p. 84]). On the other hand, we have $\text{Lat } U_\alpha \oplus \text{Lat}(U_\alpha \oplus W) = \text{Lat } Z$ (cf. [2, Lemma 1]). Hence we infer that $\mathcal{E} \subseteq K \oplus (L_2^2 \oplus \Delta L_2^2)$. Therefore $\mathcal{M}_2 \oplus \mathcal{M}_3 \subseteq \mathcal{E} \cap (L \oplus K \oplus H) \subseteq K \oplus H$. Along the same line, an even simpler argument can be applied to $\mathcal{M}_1$ and shows that $\mathcal{M}_1 \subseteq L$. Thus we have $\text{Alg } U_\alpha \oplus \text{Alg}(U_\alpha \oplus T) = \text{Alg } S$.

If $V \in \text{Alg Lat } S$, then $V = V_1 \oplus V_2$ where $V_1 \in \text{Alg Lat } U_\alpha = \text{Alg } U_\alpha$ and $V_2 \in \text{Alg Lat}(U_\alpha \oplus T) = \text{Alg}(U_\alpha \oplus T)$ by Lemma 2. From above we conclude that $V \in \text{Alg } S$ whence $S$ is reflexive. Since $(U_\alpha)^{\prime\prime} = \text{Alg } U_\alpha$ (cf. [8]) and $(U_\alpha)^{\prime\prime} \oplus (U_\alpha \oplus T)^{\prime\prime} = \{S\}^{\prime\prime}$ (cf. [1, Proposition 1.3]), Lemma 2 implies that $\{S\}^{\prime\prime} = \text{Alg } S$ if and only if $E_1 \cup F_1 \neq C$ a.e.

REFERENCES


DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHIAO TUNG UNIVERSITY, HSINCHU, TAIWAN, REPUBLIC OF CHINA