ABSOLUTE RIESZ SUMMABILITY OF FOURIER SERIES. I

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ABSTRACT. In this paper we prove some theorems on the absolute summability of Fourier series which connect diverse \( |C, \gamma| \) results such as Bosanquet's classical theorem (1936), Mohanty (1952), and Ray (1970) and the recent \( |R, \exp((\log \omega)^{\beta+1}), \gamma| \) result of Nayak (1971).

It is also shown that in some sense some of the conclusions of the paper are the best possible.

1.1. A series \( \sum U_n \) is said to be absolutely summable by the Riesz method of 'type' \( \exp((\log \omega)^{\beta+1}), \beta > 0 \), and 'order' \( r, r > 0 \), and written

\[
\sum U_n \in |R, \exp((\log \omega)^{\beta+1}), r|,
\]

if, with \( e(\omega) \equiv \exp((\log \omega)^{\beta+1}) \),

\[
\int_A \frac{e'(\omega)}{(e(\omega))^{\gamma+1}} \sum_{n<\omega} \{e(\omega) - e(n)\}^{\gamma-1}e(n)U_n \, d\omega < \infty,
\]

where \( A \) is some positive constant. In the case \( \beta = 0 \), the method \( |R, e(\omega), r| \equiv |R, \omega, r| \) is known to be equivalent to the Cesàro method \( |C, r| \).

1.2. Suppose \( f \in L(-\pi, \pi) \) where \( f \) is a \( 2\pi \)-periodic function and let

\[
f(x) \sim \frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum A_n(x).
\]

We assume, as we may, that \( a_0 = 0 = A_0 \).

We use the following notations:

\( \phi(t) = \frac{1}{2} (f(x + t) + f(x - t)) - s \), for suitably chosen \( s = s(x) \),

\( e(\omega) = \exp((\log \omega)^{\beta+1}), \beta > 0 \),

\( F(\omega, t) = \sum_{n<\omega} (e(\omega) - e(n))^{\gamma-1}e(n)n^{-1}(\log n)^{\delta} \sin nt \),

\( Q(\omega) = (e(\omega) - e(m))^{\gamma-1}e(m)m^{-1}(\log m)^{\delta} \),

where \( m \) is an integer such that \( m < \omega < m + 1 \).

Unless otherwise specified, in what follows we use "\( \sum \)" to denote "\( \sum_{n=2}^{\infty} \)" and also \( \sum_{n<\omega} \) to denote \( \sum_{n=2}^{[m]} \), \( k \) is a suitable constant, while \( K, K_1, K_2, \ldots \) denote absolute constants, possibly different at each occurrence.

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2.1. Theorems.

THEOREM I. Let $\alpha, \beta, \delta$ and $\eta$ be real numbers such that $\alpha > 0$ and either (i) $\beta = 0 = \eta, \delta < 0$, or (ii) $\delta < \eta$ and $\beta$ is any positive number however large. Then

$$
\phi(t) \left( \log \left( \frac{k}{t} \right) \right)^{\eta} \in BV(0, \pi) \Rightarrow \sum A_n(x)(\log n)^{\delta} \in |R, \exp(\log \omega)^{\beta+1}, \alpha|.
$$

The result is the best possible in the sense that $\delta < \eta$ may not be replaced by $\delta < \eta$ if $\beta \neq 0, \eta \neq 0$. For, take $k = 2\pi$ and consider $\phi(t) = (\log(2\pi/t))^{-\eta}$. Then $A_n(x) \sim (-\eta)n^{-1}(\log n)^{-\eta-1}$ (see Lemma 1, below), and $\sum A_n(x)(\log n)^{\eta} \approx -\gamma \sum (\log(n)^{\eta})^{-1}$ cannot be summable by a totally regular method. However in our next theorem, we show that if we strengthen somewhat the requirement on the function $\phi$ we can (in the case $\beta = 0$) allow $\delta = \eta$.

THEOREM II. Let $\gamma > 0, \beta > 0$ and $\delta$ and $\eta$ be real numbers such that (i) $\delta < \eta - \beta\gamma/(\gamma + 1)$, when $0 < \gamma < 1$, or (ii) $\delta < \eta - \beta/2$ when $\gamma > 1$. Suppose that $\int_0^1 |(\log(k/t))^{\eta}| d|\phi(t)| < \infty$. Then

$$
\sum A_n(x)(\log n)^{\delta} \in |R, \exp(\log \omega)^{\beta+1}, \gamma|.
$$

2.2. Remarks. (i) The case $\beta = 0$ and $\eta = 0$ of either of the two theorems contains the basic result of Bosanquet [1] for summability $|C, \alpha|, \alpha > 0$.

(ii) The case $\eta = -1$ of Theorem I improves upon Theorem A below, due to Nayak. Nayak discussed only the case $\alpha = 1$ and even there he had to take $\delta = -((\beta/2) + 1)$ and not just $\delta < -1$:

THEOREM A [6]. Let $\beta > 0$. If $\phi(t)/\log(k/t) \in BV(0, \pi)$, then

$$
\sum A_n(x)(\log n)^{-(\beta/2 + 1)} \in |R, e(\omega), 1|.
$$

(iii) Theorem II extends a theorem due to Ray [7, Theorem 1] on Cesàro summability of Fourier series. Ray’s result corresponds to the case $\beta = 0, \eta = \delta < 0$.

(iv) The case $\beta = 0, \eta = 1$ of Theorem II contains a result of Mohanty [5].

(v) We note that Theorem II furnishes a better result than Theorem I only when $\beta = 0$ (cf. §3.2, the proof of Theorem I).

(vi) We are thankful to the referee for pointing out that it follows from inequality (10) below that the hypothesis $\phi(t)(\log(k/t))^{\eta} \in BV(0, \pi)$ implies the stronger result $\sum A_n(x)(\log n)^{\delta}$ is absolutely convergent for $\delta < -1$ and $\delta < \eta - 1$. Note however that $\delta < -1$ may not be replaced by $\delta < -1$. For, consider the even, $2\pi$-periodic function $\phi$ defined by $\phi(t) = (\pi/2)\chi_{[\pi/2, \pi]}$ for $t \in [0, \pi]$. Then

$$
\phi(t) \sim \frac{\pi}{4} + \sum \frac{(-1)^n \cos(2n - 1)t}{2n - 1} \equiv \sum A_n(t).
$$

For this series we find that $\sum A_n(0)(\log n)^{-\delta}$ is not absolutely convergent if $\delta = -1$, while $(\log(k/t))^{\eta}\phi(t) \in BV(0, \pi)$ for any $\eta$.
23. For the proof of Theorem I we first prove the following theorem:

**Theorem III.** Let $\beta, \gamma, \delta$ and $\eta$ be real numbers with $\gamma > 0$. Suppose $\beta > 0$ unless $\eta = 0$ and then take $\beta > 0$; and let either (i) $\delta \leq \eta - \beta \gamma / (\gamma + 1)$ when $0 < \gamma < 1$, or (ii) $\delta \leq \eta - \beta / 2$, when $\gamma \geq 1$. Then

$$\phi(t)(\log(k/t))^\eta \in BV(0, \infty) \Rightarrow \sum A_n(x)(\log n)^\delta \in \mathbb{R}, \exp(\log \omega)^{\beta+1}, \gamma].$$

2.4. We need the following lemmas:

**Lemma 1** (see [7]). $\int_0^\infty \log((2\pi/t))^b \cos nt \, dt \sim (\pi/2) b n^{-1}(\log n)^{b-1}$, for all real $b$.

**Lemma 2.** Let $\gamma > 0$, $\beta > 0$ and also $\delta > 0$ if $\beta = 0$. Let $m$ be an integer such that $m < \omega < m + 1$. For $\omega \to \infty$ and $0 < t < \pi$, we have

$$|F(\omega, t)| \leq \begin{cases} \kappa_1 \log \omega)^{\delta-\beta} e^\gamma(\omega) + \kappa_2 (\log \omega)^{\beta+1} \log \omega, & \text{for } 0 < \gamma < 1, \\ \kappa_1 \log \omega)^{\delta-\beta} e^\gamma(\omega), & \text{for } \gamma \geq 1; \end{cases}$$

and for $\omega > (k/t) + 2$ and $0 < \gamma < 1$

$$|F(\omega, t)| \leq K t^{-\omega - \gamma} e^\gamma(\omega)(\log \omega)^{\beta+1} \left\{ (\log \omega)^{\delta-\beta} + (\log(\omega - k/t))^{\delta-\beta} \right\} + Q(\omega)$$

(iii)

and for $\omega > (k/t) + 2$ and $\gamma > 1$

$$|F(\omega, t)| \leq K t^{-\omega - \gamma} e^\gamma(\omega)(\log \omega)^{\beta+1}.$$ (2)(ii)

**Proof of Lemma 2.** Let $V(x) = V(\omega, x) = (e(\omega) - e(x))^\gamma$. Then

$$|F(\omega, t)| \leq t \sum_{n<\omega} V(n) e(n)(\log n)^{\beta+1}.$$ (2)

Take $N$ an integer such that $e(u)(\log u)^\delta$ is increasing for $u > N$. For $\gamma > 1$ and $m > N$

$$\sum_{n=N}^m V(n) e(n)(\log n)^{\beta+1}$$

$$= - \int_N^m \left( \sum_{n<x} e(n)(\log n)^{\beta+1} \right) V'(x) \, dx + V(m) \sum_{n=N}^m e(n)(\log n)^{\beta+1}$$

$$< - \int_N^m V'(x) \int_N^x e(u)(\log u)^{\delta+1} \, du \, dx - \int_N^m e(x)(\log x)^{\delta+1} V'(x) \, dx$$

$$+ V(m) \int_N^m e(x)(\log x)^{\delta+1} \, dx + V(m) e(m)(\log m)^{\beta+1}$$

$$+ \int_N^m V(x) e(x)(\log x)^{\beta+1} \, dx + V(N) e(N)(\log N)^{\beta+1}$$

$$+ \int_N^m V(x) e(x)(\log x)^{\beta+1} \left\{ \frac{1}{x} \left( (\beta + 1)(\log x)^{\beta} + \delta(\log x)^{-1} \right) \right\} \, dx,$$

on integrating by parts,

$$< K_1 \int_N^\omega (e(\omega) - e(x))^\gamma \frac{e(x)(\beta + 1)(\log x)}{x} \cdot x(\log x)^{\beta-\delta} \, dx$$

$$+ K_2 e^\gamma(\omega)$$

$$< K_\omega(\log \omega)^{\delta-\beta} e^\gamma(\omega).$$
For $0 < \gamma < 1$, and $m > N$ we get 
\[ \sum_{n=N}^{m} V(n)e(n)(\log n)^\delta < \int_{\omega}^{\omega' \delta} \{ e(\omega) - e(x) \}^{-1} e(x)(\log x)^\delta \, dx + V(m)e(m)(\log m)^\delta \]
\[ < K\omega(\log \omega)^{\delta - \beta} e^\gamma(\omega) + mQ(\omega). \]

Also $\sum_{n<N} V(n)e(n)(\log n)^\delta < Ke^{\gamma-1}(\omega)$. This proves (1). Note that when $\gamma > 1$, the second term in (1) is dominated by the first.

To prove (2) we first take up the case $0 < \gamma < 1$. Let $\omega_1 = [\omega - k/t]$ and $N$ be an integer such that the functions (i) $e(x)x^{-1}(\log x)^\delta$, (ii) $e'(x)$ and (iii) $(e'(x))^2(\log x)^{\delta - \beta}$ are nondecreasing for $x > N$. Let 
\[ F(\omega, t) = \sum_{n=2}^{N} + \sum_{N+1}^{\omega_1} + \sum_{\omega_1+1}^{m} = S_1 + S_2 + S_3, \text{ say.} \]

In the case $\omega_1 < N$, $S_2$ is zero and the lower limit in $S_3$ is $(N + 1)$.

Now 
\[ |S_1| < K\{ e(\omega) - e(N) \}^{-1} < Ke^{\gamma-1}(\omega), \]
\[ |S_2| < K\{ e(\omega) - e(\omega_1) \}^{-1} e(\omega_1)\omega_1^{-1}(\log \omega_1)^\delta \max_{N < a < b < \omega} \left| \sum_{a}^{b} \sin nt \right| \]
\[ < Kr^{-1}\{ (\omega - \omega_1)e'(\omega^\ast) \}^{-1} e(\omega_1)\omega_1^{-1}(\log \omega_1)^\delta \]
\[ < Kr^{-1}t^{-1}(\gamma-1)\{ e'(\omega_1) \}^{-1} e(\omega_1)\omega_1^{-1}(\log \omega_1)^\delta \]
\[ < Kr^{-1}e^{-1}(\omega)(\log \omega)^{\beta(\gamma-1)+\delta}. \quad (3) \]

If $\delta > \beta$,
\[ |S_3| < \int_{\omega_1}^{\omega} \{ e(\omega) - e(u) \}^{-1} e(u)u^{-1}(\log u)^\delta \, du \]
\[ + \{ e(\omega) - e(m) \}^{-1} e(m)m^{-1}(\log m)^\delta \]
\[ < K(\log \omega)^{\delta - \beta} \{ e(\omega) - e(\omega_1) \} \gamma + Q(\omega) \]
\[ < Kr^{-1}e^{-1}(\omega)(\log \omega)^{\delta + \beta(\gamma-1)} + Q(\omega), \quad (4) \]

and if $\delta < \beta$,
\[ |S_3| < K(\log(\omega - k/t))^{\delta - \beta} \{ e(\omega) - e(\omega_1) \} \gamma \]
\[ + \{ e(\omega) - e(m) \}^{-1} e(m)m^{-1}(\log m)^\delta \]
\[ < Kr^{-1}(\log(\omega - k/t))^{\delta - \beta} e^\gamma(\omega) + Q(\omega). \quad (5) \]

When $\gamma > 1$ obviously
\[ |F(\omega, t)| < Kr^{-1}\{ e(\omega) - e(2) \}^{-1} e(\omega)\omega^{-1}(\log \omega)^\delta. \quad (6) \]

Collecting the results from (3) to (6) completes the proof of the lemma.
Lemma 3. Suppose that $\gamma > 0$ and $c > 0$. If $\sum U_n \in |C, \gamma|$ then

$$\sum U_n (\log n)^c \in |C, \gamma|.$$ 

This is a particular case of the much more general Anderson-Bosanquet-Chow-Peyerimhoff Theorem [2] on summability factors of absolute Cesàro summability.

Lemma 4 [7]. Let $\alpha > 0$ and $\eta < 0$. If (i) $\phi(t)(\log(k/t))^\eta \in BV(0, \pi)$, and (ii) $\int_0^\pi |\phi(t)|/t(\log(k/t))^{1-\eta} \, dt < \infty$ then $\sum A_n(x)(\log n)^\eta$ is summable $|C, \alpha|$.

Lemma 5. Let $\eta < 0$. Then (i) and (ii) hold if and only if (iii) holds, where

(i) $\phi(t)(\log(k/t))^\eta \in BV(0, \pi)$,

(ii) $(\log(k/t))^{\eta-1}\phi(t)/t \in L(0, \pi)$, and

(iii) $\int_0^\pi (\log(k/t))^\eta |d\phi(t)| < \infty$.

Proof. Since

$$d \{ (\log(k/t))^\eta \phi(t) \} = (\log(k/t))^\eta d\phi(t) - \eta(\log(k/t))^{\eta-1}\phi(t) dt/t \quad (*)$$

then (iii) follows from (i) and (ii).

Conversely, we have, for $0 < t < \pi$, $|\phi(t)| < |\phi(\pi)| + \int_0^\pi |d\phi(u)|$. Hence

$$\int_0^\pi (\log(k/t))^{\eta-1} \frac{\phi(t)}{t} \, dt$$

$$\leq |\phi(\pi)| \int_0^\pi (\log(k/t))^{\eta-1} \frac{dt}{t} + \int_0^\pi \int_0^\pi |d\phi(u)| (\log(k/t))^{\eta-1} \frac{dt}{t}$$

$$\leq K + \int_0^\pi \int_0^u (\log(k/t))^{\eta-1} \frac{dt}{t}|d\phi(u)|$$

$$= K + \frac{1}{|\eta|} \int_0^\pi (\log(k/u))^\eta |d\phi(u)|.$$ 

Thus (ii) follows (iii), and then (i) follows from (ii) and (iii) with the aid of (*).

3.1. Proof of Theorems. We shall combine the proofs for Theorems II and III.

For Theorem II in the case $\beta = 0, \delta = \eta < 0$, the result follows from Lemmas 4 and 5, and then for $\beta = 0, 0 < \delta < \eta < 0$, the result follows by Lemma 3. Similarly for $\beta = 0, \delta < 0 < \eta$, the result follows from the case $\beta = 0, 0 \leq \delta < \eta$, by Lemma 3. Hence it remains to consider the cases (i) $\beta = 0, \eta > \delta > 0$ and (ii) $\beta > 0$. Similarly, for Theorem III the case $\beta = 0 = \eta, \delta < 0$, follows from the case $\beta = 0, \eta = 0 = \delta$, after Lemma 3. Hence we need to consider this theorem also only for the cases (iii) $\beta = 0 = \eta = \delta$ and (iv) $\beta > 0$. 

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To begin with for Theorem III, we note that, for \( n > 1, \)
\[
\frac{\pi}{2} A_n(x) = \int_0^\pi \phi(t) \cos nt \, dt
\]
\[
= \left[ \phi(t)(\log(k/t))^n \int_0^t (\log(k/u))^{-n} \cos nu \, du \right]_0^\pi
\]
\[
- \int_0^\pi \left\{ \int_0^t (\log(k/u))^{-n} \cos nu \, du \right\} d\{\phi(t)(\log(k/t))^n\}
\]
\[
= \phi(\pi)(\log(k/\pi))^n \int_0^\pi (\log(k/u))^{-n} \cos nu \, du
\]
\[
- \frac{1}{n} \int_0^\pi \left\{ \sin nt(\log(k/t))^{-n} - \eta \int_0^t \frac{\sin nu}{u} (\log(k/u))^{-n-1} \, du \right\}
\cdot d\{\phi(t)(\log(k/t))^n\}. \quad (7)
\]

For \( t > 1/n \) and \( \eta \neq 0 \)
\[
\left| \int_0^t \sin nu(\log(k/u))^{-n-1} u^{-1} \, du \right| = \left| \int_0^{1/n} + \int_{1/n}^t \right|
\]
\[
< n \int_0^{1/n} (\log(k/u))^{-n-1} \, du + \int_{1/n}^t \left[ -\cos nu(\log(k/u))^{-n-1} u^{-1} \right] \, du
\]
\[
+ \frac{1}{n} \int_{1/n}^t \cos nu \, u^{-1}(\log(k/u))^{-n-1} \, du
\]
\[
< K(\log n)^{-n-1}, \quad (8)
\]

and for \( 0 < t < 1/n \) and \( \eta \neq 0 \)
\[
\left| \int_0^t u^{-1}(\log(k/u))^{-n-1} \sin nu \, du \right| < n \int_0^{1/n} (\log(k/u))^{-n-1} \, du
\]
\[
< K(\log n)^{-n-1}. \quad (9)
\]

Thus, since \( \int_0^\pi d\{\phi(t)(\log(k/t))^n\} \) is finite, from (7)–(9) and Lemma 1, we get
\[
\left| \frac{\pi}{2} A_n(x) + \frac{1}{n} \int_0^\pi \sin nt(\log(k/t))^{-n} d\{\phi(t)(\log(k/t))^n\} \right| < K n^{-1}(\log n)^{-n-1}. \quad (10)
\]

For \( \eta = 0 \), we see by (7), the right-hand side in (10) is zero.

Since \( \sum n^{-1}(\log n)^{\delta-\eta-1} \), \( \eta - \delta > 0 \), is absolutely convergent, to prove that
\( \sum A_n(x)(\log n)^{\delta} \in \mathbb{R}, e(\omega), \gamma \) (whether \( \eta = 0 \) or not), it is enough to show that the integral
\[
\int_2^\infty (\log \omega)^\delta \omega^{-1} e^{-\gamma(\omega)} \left| \sum_{n<\omega} \{e(\omega) - e(n)\} \gamma^{-1} e(n)(\log n)^{\delta} n^{-1} \right|
\]
\[
\int_0^\pi \sin nt(\log(k/t))^{-n} d\{\phi(t)(\log(k/t))^n\} \, d\omega
\]
is convergent. As \( \phi(t)(\log(k/t))^n \in BV(0, \pi) \), it is sufficient to show that
\[
\int_2^\infty (\log \omega)^\delta \omega^{-1} e^{-\gamma(\omega)} \left| F(\omega, t) \right| \, d\omega = O((\log(k/t))^\eta), \quad (11)
\]
uniformly in $t$, $0 < t < \pi$. Let
\[ I(t) = \int_2^\infty (\log \omega)^\beta \omega^{-1} e^{-\gamma(\omega)} |F(\omega, t)| \, d\omega. \tag{12} \]

For the proof for Theorem II we note that, for $n > 1$,
\[ \frac{\pi}{2} A_n(x) = \int_0^\pi \phi(t) \cos nt \, dt = -(1/n) \int_0^\pi \sin nt \, d\phi(t). \]

As $\int_0^\pi (\log(k/t))^\gamma \, |d\phi(t)|$ is finite, in this case also it is sufficient to show that the requirement (11) is fulfilled, that is
\[ I(t) = O((\log(k/t))^\gamma), \quad \text{uniformly in } t, 0 < t < \pi. \tag{13} \]

Let $r = (k/t)(\log(k/t))^{\beta'} + 2$, where $\gamma' = \min(\gamma/(\gamma + 1), 1/2)$, and
\[ I(t) = \int_2^r + \int_r^\infty = I_1(t) + I_2(t), \quad \text{say.} \tag{14} \]

After the estimates in (1) we get
\[
\begin{align*}
I_1(t) &< K_1 t \int_2^r (\log \omega)^\delta \, d\omega + K_2 t \int_2^r (\log \omega)^\delta \omega^{-1} e^{-\gamma(\omega)} mQ(\omega) \, d\omega \\
&< K_1 t \tau (\log \tau)^\delta + K_2 t \tau^{-1} (\log \tau)^{\delta + \beta \gamma} \\
&= O((\log(k/t))^\gamma), \quad \text{uniformly in } t, 0 < t < \pi, \tag{15}
\end{align*}
\]
since, for $0 < \gamma < 1$,
\[
\begin{align*}
\int_2^r (\log \omega)^\delta \omega^{-1} e^{-\gamma(\omega)} mQ(\omega) \, d\omega \\
&< \sum_{m < \tau} \int_m^{m+1} (e(\omega) - e(m))^{\gamma-1} (\log \omega)^\beta \omega^{-\gamma(\omega)} e(m) \cdot (\log m)^\delta \, d\omega \\
&< K \sum_{m < \tau} (\log m)^\delta e^{-\gamma(m+1)} [e(m+1) - e(m)]^\gamma \\
&< K \sum_{m < \tau} m^{-\gamma} (\log m)^{\delta + \beta \gamma}. \tag{16}
\end{align*}
\]

For $I_2(t)$, we first consider the case $0 < \gamma < 1$. Using the order estimates from (2)(i) and proceeding as for (15) and (16),
\[
\begin{align*}
I_2(t) &< K_1 t^{-\gamma} \int_\tau^\infty (\log \omega)^{\beta(\gamma+1)} \omega^{-1} \{ (\log \omega)^{\delta-\beta} + (\log(\omega - k/t))^{\delta-\beta} \} \, d\omega \\
&\quad + K_2 \int_\tau^\infty (\log \omega)^\delta \omega^{-1} e^{-\gamma(\omega)} Q(\omega) \, d\omega \\
&< K_1 t^{-\gamma} (\log \tau)^{\beta(\gamma+1)} \{ (\log \tau)^{\delta-\beta} + (\log(\tau - k/t))^{\delta-\beta} \} \\
&\quad + K_2 \sum_{m < \tau} m^{-\gamma-1} (\log m)^{\delta + \beta \gamma} \\
&= O((\log(k/t))^\gamma), \quad \text{uniformly in } t, 0 < t < \pi. \tag{17}
\end{align*}
\]
For $\gamma > 1$, the order estimates in (2)(ii) give
\[
I_2(t) < Kt^{-1}\int_t^\infty (\log \omega)\beta \omega^{-2} d\omega
= O((\log(k/t))^\gamma), \quad \text{uniformly in } t, 0 < t < \pi.
\] (18)
This completes the proof for Theorems II and III.

3.2. Proof of Theorem I. Note that $\delta < \eta$ implies that $\delta < \eta - \beta \gamma / (\gamma + 1)$, for all arbitrarily small positive $\gamma$, $0 < \gamma < 1$. We choose $\gamma$ such that it is also less than $\alpha$. Now by Theorem III, $\sum A_n(x)(\log n)^\delta \in |R, e(\omega), \gamma|$. Hence, by the First Theorem of Consistency for absolute Riesz summability (see [3, §1.9])
\[
\sum A_n(x)(\log n)^\delta \in |R, e(\omega), \alpha|.
\]

4. We obtain the following corollary to our Theorems I and II.

**Corollary.** Let $\alpha > 0$ and $\delta$ and $\eta$ be real numbers. If either
(I) $\eta \neq 0$, $\delta < \eta$ and $\phi(t)(\log(k/t))^\eta \in BV(0, \pi)$, or
(II) $0 < \eta$, $\delta < \eta$ and $\int_0^\infty (\log(k/t))^\eta |d\phi(t)|$ is finite
then $\sum A_n(x)(\log n)^\delta \in |C, \alpha|$.

Part (I) of the corollary follows from Theorem I after a ‘Second Theorem of Consistency’ for absolute Riesz summability (see [4, Theorem 2]).

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**References**