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ABSOLUTE RIESZ SUMMABILITY OF FOURIER SERIES. I

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ABSTRACT. In this paper we prove some theorems on the absolute summability of Fourier series which connect diverse $|C, \gamma|$ results such as Bosanquet's classical theorem (1936), Mohanty (1952), and Ray (1970) and the recent $|R, \exp((\log \omega)^{\beta+1}), \gamma|$ result of Nayak (1971).

It is also shown that in some sense some of the conclusions of the paper are the best possible.

1.1. A series $\sum U_n$ is said to be absolutely summable by the Riesz method of 'type' $\exp((\log \omega)^{\beta+1})$, $\beta \geqslant 0$, and 'order' r, $r > 0$, and written

$$\sum U_n \in |R, \exp((\log \omega)^{\beta+1}), r|,$$

if, with $e(\omega) \equiv \exp((\log \omega)^{\beta+1})$,

$$\int_A^\infty \frac{e'(\omega)}{(e(\omega))^{r+1}} \left| \sum_{n<\omega} \{e(\omega) - e(n)\}^{r-1} e(n) U_n \right| d\omega < \infty,$$

where A is some positive constant. In the case $\beta = 0$, the method $|R, e(\omega), r| \equiv |R, \omega, r|$ is known to be equivalent to the Cesàro method $|C, r|$.

1.2. Suppose $f \in L(-\pi, \pi)$ where f is a 2π-periodic function and let

$$f(x) \sim \frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \equiv \sum A_n(x).$$

We assume, as we may, that $a_0 = 0 = A_0$.

We use the following notations:

$\phi(t) = \frac{1}{2}\{f(x + t) + f(x - t)\} - s$, for suitably chosen $s = s(x)$,

$e(\omega) = \exp\{(\log \omega)^{\beta+1}\}$, $\beta \geqslant 0$,

$F(\omega, t) = \sum_{n<\omega}\{e(\omega) - e(n)\}^{\gamma-1} e(n) n^{-1} (\log n)^\delta \sin nt$,

$Q(\omega) = \{e(\omega) - e(m)\}^{\gamma-1} e(m) m^{-1} (\log m)^\delta$, where m is an integer such that $m < \omega \leqslant m + 1$.

Unless otherwise specified, in what follows we use "Σ" to denote "$\Sigma_{n=2}^\infty$" and also $\Sigma_{n<\omega}$ to denote $\Sigma_{n=2}^{[m]}$. k is a suitable constant, while $K, K_1, K_2, \ldots$ denote absolute constants, possibly different at each occurrence.

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2.1. Theorems.

THEOREM I. *Let α, β, δ and η be real numbers such that $\alpha > 0$ and either* (i) $\beta = 0 = \eta$, $\delta < 0$, *or* (ii) $\delta < \eta$ *and β is any positive number however large. Then*

$$\phi(t)\left(\log\frac{k}{t}\right)^{\eta} \in BV(0, \pi) \Rightarrow \sum A_n(x)(\log n)^{\delta} \in |R, \exp(\log \omega)^{\beta+1}, \alpha|.$$

The result is the best possible in the sense that '$\delta < \eta$' may not be replaced by '$\delta \leqslant \eta$' if $\beta \neq 0$, $\eta \neq 0$. For, take $k = 2\pi$ and consider $\phi(t) = (\log(2\pi/t))^{-\eta}$. Then $A_n(x) \sim (-\eta)n^{-1}(\log n)^{-\eta-1}$ (see Lemma 1, below), and $\sum A_n(x)(\log n)^{\eta} \approx -\eta\sum(n(\log n))^{-1}$ cannot be summable by a totally regular method. However in our next theorem, we show that if we strengthen somewhat the requirement on the function ϕ we can (in the case $\beta = 0$) allow $\delta = \eta$.

THEOREM II. *Let $\gamma > 0$, $\beta \geqslant 0$ and δ and η be real numbers such that* (i) $\delta \leqslant \eta - \beta\gamma/(\gamma + 1)$, *when* $0 < \gamma < 1$, *or* (ii) $\delta \leqslant \eta - \beta/2$ *when* $\gamma \geqslant 1$. *Suppose that* $\int_0^{\pi}(\log(k/t))^{\eta}|d\phi(t)| < \infty$. *Then*

$$\sum A_n(x)(\log n)^{\delta} \in |R, \exp(\log \omega)^{\beta+1}, \gamma|.$$

2.2. Remarks. (i) The case $\beta = 0$ and $\eta = 0$ of either of the two theorems contains the basic result of Bosanquet [1] for summability $|C, \alpha|$, $\alpha > 0$.

(ii) The case $\eta = -1$ of Theorem I improves upon Theorem A below, due to Nayak. Nayak discussed only the case $\alpha = 1$ and even there he had to take $\delta = -((\beta/2) + 1)$ and not just '$\delta < -1$':

THEOREM A [6]. *Let $\beta > 0$. If $\phi(t)/\log(k/t) \in BV(0, \pi)$, then*

$$\sum A_n(x)(\log n)^{-((\beta/2)+1)} \in |R, e(\omega), 1|.$$

(iii) Theorem II extends a theorem due to Ray [7, Theorem 1] on Cesàro summability of Fourier series. Ray's result corresponds to the case $\beta = 0$, $\eta = \delta < 0$.

(iv) The case $\beta = 0$, $\eta = 1$ of Theorem II contains a result of Mohanty [5].

(v) We note that Theorem II furnishes a better result than Theorem I only when $\beta = 0$ (cf. §3.2, the proof of Theorem I).

(vi) We are thankful to the referee for pointing out that it follows from inequality (10) below that the hypothesis $\phi(t)(\log(k/t))^{\eta} \in BV(0, \pi)$ implies the stronger result $\sum A_n(x)(\log n)^{\delta}$ is absolutely convergent for $\delta < -1$ and $\delta \leqslant \eta - 1$. Note however that $\delta < -1$ may not be replaced by $\delta \leqslant -1$. For, consider the even, 2π-periodic function ϕ defined by $\phi(t) = (\pi/2)\chi_{[\pi/2,\pi]}$ for $t \in [0, \pi]$. Then

$$\phi(t) \sim \frac{\pi}{4} + \sum_1^{\infty} \frac{(-1)^n \cos(2n-1)t}{2n-1} \equiv \sum_0^{\infty} A_\nu(t).$$

For this series we find that $\sum A_n(0)(\log n)^{-\delta}$ is not absolutely convergent if $\delta = -1$, while $(\log(k/t))^{\eta}\phi(t) \in BV(0, \pi)$ for any η.

2.3. For the proof of Theorem I we first prove the following theorem:

THEOREM III. *Let β, γ, δ and η be real numbers with $\gamma > 0$. Suppose $\beta > 0$ unless $\eta = 0$ and then take $\beta \geqslant 0$; and let either* (i) $\delta \leqslant \eta - \beta\gamma/(\gamma + 1)$ *when* $0 < \gamma < 1$, *or* (ii) $\delta \leqslant \eta - \beta/2$, *when* $\gamma \geqslant 1$. *Then*

$$\phi(t)(\log(k/t))^{\eta} \in BV(0, \pi) \Rightarrow \sum A_n(x)(\log n)^{\delta} \in |R, \exp(\log \omega)^{\beta+1}, \gamma|.$$

2.4. We need the following lemmas:

LEMMA 1 (SEE [7]). $\int_0^{\pi}(\log(2\pi/t))^b \cos nt\, dt \sim (\pi/2)bn^{-1}(\log n)^{b-1}$, *for all real b.*

LEMMA 2. *Let $\gamma > 0$, $\beta \geqslant 0$ and also $\delta \geqslant 0$ if $\beta = 0$. Let m be an integer such that $m < \omega \leqslant m + 1$. For $\omega \to \infty$ and $0 < t \leqslant \pi$, we have*

$$|F(\omega, t)| \leqslant \begin{cases} Kt\omega(\log \omega)^{\delta-\beta}e^{\gamma}(\omega) + tmQ(\omega), & \text{for } 0 < \gamma < 1, \\ Kt\omega(\log \omega)^{\delta-\beta}e^{\gamma}(\omega), & \text{for } \gamma > 1; \end{cases} \tag{1}$$

and for $\omega > (k/t) + 2$ and $0 < \gamma < 1$

$$|F(\omega, t)| \leqslant Kt^{-\gamma}\omega^{-\gamma}e^{\gamma}(\omega)(\log \omega)^{\beta\gamma}\{(\log \omega)^{\delta-\beta} + (\log(\omega - k/t))^{\delta-\beta}\} + Q(\omega) \tag{2(i)}$$

and for $\omega > (k/t) + 2$ and $\gamma \geqslant 1$

$$|F(\omega, t)| \leqslant Kt^{-1}\omega^{-1}e^{\gamma}(\omega)(\log \omega)^{\delta}. \tag{2(ii)}$$

PROOF OF LEMMA 2. Let $V(x) \equiv V(\omega, x) = \{e(\omega) - e(x)\}^{\gamma-1}$. Then

$$|F(\omega, t)| \leqslant t \sum_{n<\omega} V(n)e(n)(\log n)^{\delta}.$$

Take N an integer such that $e(u)(\log u)^{\delta}$ is increasing for $u \geqslant N$. For $\gamma > 1$ and $m > N$

$$\begin{aligned}
&\sum_{n=N}^{m} V(n)e(n)(\log n)^{\delta} \\
&\quad = -\int_N^m \Big(\sum_{n<x} e(n)(\log n)^{\delta}\Big)V'(x)\, dx + V(m)\sum_{n=N}^{m} e(n)(\log n)^{\delta} \\
&\quad \leqslant -\int_N^m V'(x)\int_N^x e(u)(\log u)^{\delta}\, du\, dx - \int_N^m e(x)(\log x)^{\delta}V'(x)\, dx \\
&\qquad + V(m)\int_N^m e(x)(\log x)^{\delta}\, dx + V(m)e(m)(\log m)^{\delta} \\
&\quad = \int_N^m V(x)e(x)(\log x)^{\delta}\, dx + V(N)e(N)(\log N)^{\delta} \\
&\qquad + \int_N^m V(x)e(x)(\log x)^{\delta}\Big(\frac{1}{x}\{(\beta + 1)(\log x)^{\beta} + \delta(\log x)^{-1}\}\Big)\, dx,
\end{aligned}$$

on integrating by parts,

$$\begin{aligned}
&\leqslant K_1\int_N^{\omega}\{e(\omega) - e(x)\}^{\gamma-1}\frac{e(x)(\beta + 1)(\log x)^{\beta}}{x}\cdot x(\log x)^{\delta-\beta}\, dx \\
&\qquad + K_2e^{\gamma-1}(\omega) \\
&\leqslant K\omega(\log \omega)^{\delta-\beta}e^{\gamma}(\omega).
\end{aligned}$$

For $0 < \gamma < 1$, and $m > N$ we get

$$\sum_{n=N}^{m} V(n)e(n)(\log n)^{\delta}$$
$$< \int_N^{\omega} \{e(\omega) - e(x)\}^{\gamma-1} e(x)(\log x)^{\delta}\, dx + V(m)e(m)(\log m)^{\delta}$$
$$< K\omega(\log \omega)^{\delta-\beta} e^{\gamma}(\omega) + mQ(\omega).$$

Also $\sum_{n<N} V(n)e(n)(\log n)^{\delta} < Ke^{\gamma-1}(\omega)$. This proves (1). Note that when $\gamma > 1$, the second term in (1) is dominated by the first.

To prove (2) we first take up the case $0 < \gamma < 1$. Let $\omega_1 = [\omega - k/t]$ and N be an integer such that the functions (i) $e(x)x^{-1}(\log x)^{\delta}$, (ii) $e'(x)$ and (iii) $(e'(x))^{\gamma}(\log x)^{\delta-\beta}$ are nondecreasing for $x > N$. Let

$$F(\omega, t) = \sum_{n=2}^{N} + \sum_{N+1}^{\omega_1} + \sum_{\omega_1+1}^{m} = S_1 + S_2 + S_3, \quad \text{say.}$$

In the case $\omega_1 < N$, S_2 is zero and the lower limit in S_3 is $(N + 1)$.

Now

$$|S_1| < K\{e(\omega) - e(N)\}^{\gamma-1} < Ke^{\gamma-1}(\omega),$$
$$|S_2| < K\{e(\omega) - e(\omega_1)\}^{\gamma-1} e(\omega_1)\omega_1^{-1}(\log \omega_1)^{\delta} \max_{N<a<b<\omega} \left| \sum_a^b \sin nt \right|$$
$$< Kt^{-1}\{(\omega - \omega_1)e'(\omega^*)\}^{\gamma-1} e(\omega_1)\omega_1^{-1}(\log \omega_1)^{\delta}$$
$$\text{for some } \omega^*, \omega^* \in (\omega_1, \omega),$$
$$< Kt^{-1}t^{-(\gamma-1)}\{e'(\omega_1)\}^{\gamma-1} e(\omega_1)\omega_1^{-1}(\log \omega_1)^{\delta}$$
$$< Kt^{-\gamma}\omega^{-\gamma}e^{\gamma}(\omega)(\log \omega)^{\beta(\gamma-1)+\delta}. \tag{3}$$

If $\delta > \beta$,

$$|S_3| < \int_{\omega_1}^{\omega} \{e(\omega) - e(u)\}^{\gamma-1} e(u)u^{-1}(\log u)^{\delta}\, du$$
$$+ \{e(\omega) - e(m)\}^{\gamma-1} e(m)m^{-1}(\log m)^{\delta}$$
$$< K(\log \omega)^{\delta-\beta}\{e(\omega) - e(\omega_1)\}^{\gamma} + Q(\omega)$$
$$< Kt^{-\gamma}\omega^{-\gamma}(\log \omega)^{\delta+\beta(\gamma-1)} e^{\gamma}(\omega) + Q(\omega), \tag{4}$$

and if $\delta < \beta$,

$$|S_3| < K(\log(\omega - k/t))^{\delta-\beta}\{e(\omega) - e(\omega_1)\}^{\gamma}$$
$$+ \{e(\omega) - e(m)\}^{\gamma-1} e(m)m^{-1}(\log m)^{\delta}$$
$$\leqslant Kt^{-\gamma}\omega^{-\gamma}(\log \omega)^{\beta\gamma}(\log(\omega - k/t))^{\delta-\beta} e^{\gamma}(\omega) + Q(\omega). \tag{5}$$

When $\gamma > 1$ obviously

$$|F(\omega, t)| < Kt^{-1}\{e(\omega) - e(2)\}^{\gamma-1} e(\omega)\omega^{-1}(\log \omega)^{\delta}. \tag{6}$$

Collecting the results from (3) to (6) completes the proof of the lemma.

LEMMA 3. *Suppose that* $\gamma \geq 0$ *and* $c > 0$. *If* $\sum U_n \in |C, \gamma|$ *then*

$$\sum U_n(\log n)^{-c} \in |C, \gamma|.$$

This is a particular case of the much more general Anderson-Bosanquet-Chow-Peyerimhoff Theorem [2] on summability factors of absolute Cesàro summability.

LEMMA 4 [7]. *Let* $\alpha > 0$ *and* $\eta < 0$. *If* (i) $\phi(t)(\log(k/t))^{\eta} \in BV(0, \pi)$, *and* (ii) $\int_0^{\pi} |\phi(t)|/t(\log(k/t))^{1-\eta}\, dt < \infty$ *then* $\sum A_n(x)(\log n)^{\eta}$ *is summable* $|C, \alpha|$.

LEMMA 5. *Let* $\eta < 0$. *Then* (i) *and* (ii) *hold if and only if* (iii) *holds, where*
(i) $\phi(t)(\log(k/t))^{\eta} \in BV(0, \pi)$,
(ii) $(\log(k/t))^{\eta-1}\phi(t)/t \in L(0, \pi)$, *and*
(iii) $\int_0^{\pi}(\log(k/t))^{\eta}\, |d\phi(t)| < \infty$.

PROOF. Since

$$d\{(\log(k/t))^{\eta}\phi(t)\} = (\log(k/t))^{\eta}d\phi(t) - \eta(\log(k/t))^{\eta-1}\phi(t)dt/t \qquad (*)$$

then (iii) follows from (i) and (ii).

Conversely, we have, for $0 < t < \pi$, $|\phi(t)| \leq |\phi(\pi)| + \int_t^{\pi}|d\phi(u)|$. Hence

$$\begin{aligned}
&\int_0^{\pi}(\log(k/t))^{\eta-1}\frac{|\phi(t)|}{t}\, dt \\
&\qquad \leq |\phi(\pi)|\int_0^{\pi}(\log(k/t))^{\eta-1}\frac{dt}{t} + \int_0^{\pi}\int_t^{\pi}|d\phi(u)|(\log(k/t))^{\eta-1}\frac{dt}{t} \\
&\qquad \leq K + \int_0^{\pi}\int_0^{u}(\log(k/t))^{\eta-1}\frac{dt}{t}|d\phi(u)| \\
&\qquad = K + \frac{1}{|\eta|}\int_0^{\pi}(\log(k/u))^{\eta}|d\phi(u)|.
\end{aligned}$$

Thus (ii) follows (iii), and then (i) follows from (ii) and (iii) with the aid of (*).

3.1. Proof of Theorems. We shall combine the proofs for Theorems II and III.

For Theorem II in the case $\beta = 0$, $\delta = \eta < 0$, the result follows from Lemmas 4 and 5, and then for $\beta = 0$, $\delta < \eta < 0$, the result follows by Lemma 3. Simiarly for $\beta = 0$, $\delta < 0 < \eta$, the result follows from the case $\beta = 0$, $0 \leq \delta \leq \eta$, by Lemma 3. Hence it remains to consider the cases (i) $\beta = 0$, $\eta \geq \delta \geq 0$ and (ii) $\beta > 0$. Similarly, for Theorem III the case $\beta = 0 = \eta$, $\delta < 0$, follows from the case $\beta = 0$, $\eta = 0 = \delta$, after Lemma 3. Hence we need to consider this theorem also only for the cases (iii) $\beta = 0 = \eta = \delta$ and (iv) $\beta > 0$.

To begin with for Theorem III, we note that, for $n \geqslant 1$,

$$\begin{aligned}\frac{\pi}{2}A_n(x) &= \int_0^\pi \phi(t)\cos nt\,dt \\ &= \left[\phi(t)(\log(k/t))^\eta \int_0^t (\log(k/u))^{-\eta}\cos nu\,du\right]_0^\pi \\ &\quad - \int_0^\pi \left\{\int_0^t (\log(k/u))^{-\eta}\cos nu\,du\right\} d\{\phi(t)(\log(k/t))^\eta\} \\ &= \phi(\pi)(\log(k/\pi))^\eta \int_0^\pi (\log(k/u))^{-\eta}\cos nu\,du \\ &\quad -\frac{1}{n}\int_0^\pi \left\{\sin nt(\log(k/t))^{-\eta} - \eta\int_0^t \frac{\sin nu}{u}(\log(k/u))^{-\eta-1}\,du\right\} \\ &\quad \cdot d\{\phi(t)(\log(k/t))^\eta\}. \end{aligned} \tag{7}$$

For $t > 1/n$ and $\eta \neq 0$

$$\begin{aligned}\left|\int_0^t \sin nu(\log(k/u))^{-\eta-1}u^{-1}\,du\right| &= \left|\int_0^{1/n} + \int_{1/n}^t\right| \\ &\leqslant n\int_0^{1/n}(\log(k/u))^{-\eta-1}\,du + \left|\left[-\cos nu(\log(k/u))^{-\eta-1}n^{-1}u^{-1}\right]_{1/n}^t\right| \\ &\quad + \frac{1}{n}\left|\int_{1/n}^t \cos nu\frac{d}{du}\{u^{-1}(\log(k/u))^{-\eta-1}\}\right| \\ &\leqslant K(\log n)^{-\eta-1}, \end{aligned} \tag{8}$$

and for $0 < t \leqslant 1/n$ and $\eta \neq 0$

$$\begin{aligned}\left|\int_0^t u^{-1}(\log(k/u))^{-\eta-1}\sin nu\,du\right| &\leqslant n\int_0^{1/n}(\log(k/u))^{-\eta-1}\,du \\ &\leqslant K(\log n)^{-\eta-1}. \end{aligned} \tag{9}$$

Thus, since $\int_0^\pi |d\{\phi(t)(\log k/t)^\eta\}|$ is finite, from (7)–(9) and Lemma 1, we get

$$\left|\frac{\pi}{2}A_n(x) + \frac{1}{n}\int_0^\pi \sin nt(\log(k/t))^{-\eta}\,d\{\phi(t)(\log(k/t))^\eta\}\right| \leqslant Kn^{-1}(\log n)^{-\eta-1}. \tag{10}$$

For $\eta = 0$, we see by (7), the right-hand side in (10) is zero.

Since $\sum n^{-1}(\log n)^{\delta-\eta-1}$, $\eta - \delta > 0$, is absolutely convergent, to prove that $\sum A_n(x)(\log n)^\delta \in |R, e(\omega), \gamma|$ (whether $\eta = 0$ or not), it is enough to show that the integral

$$\int_2^\infty (\log\omega)^\beta\omega^{-1}e^{-\gamma}(\omega)\left|\sum_{n<\omega}\{e(\omega) - e(n)\}^{\gamma-1}e(n)(\log n)^\delta n^{-1}\right.$$
$$\left.\int_0^\pi \sin nt(\log(k/t))^{-\eta}\,d\{\phi(t)(\log(k/t))^\eta\}\right|\,d\omega$$

is convergent. As $\phi(t)(\log(k/t))^\eta \in BV(0, \pi)$, it is sufficient to show that

$$\int_2^\infty (\log\omega)^\beta\omega^{-1}e^{-\gamma}(\omega)|F(\omega, t)|\,d\omega = O((\log(k/t))^\eta), \tag{11}$$

uniformly in t, $0 < t \leqslant \pi$. Let

$$I(t) = \int_2^\infty (\log \omega)^\beta \omega^{-1} e^{-\gamma}(\omega) |F(\omega, t)| \, d\omega. \tag{12}$$

For the proof for Theorem II we note that, for $n \geqslant 1$,

$$\frac{\pi}{2} A_n(x) = \int_0^\pi \phi(t) \cos nt \, dt = -(1/n) \int_0^\pi \sin nt \, d\phi(t).$$

As $\int_0^\pi (\log(k/t))^\eta \, |d\phi(t)|$ is finite, in this case also it is sufficient to show that the requirement (11) is fulfilled, that is

$$I(t) = O((\log(k/t))^\eta), \quad \text{uniformly in } t, 0 < t \leqslant \pi. \tag{13}$$

Let $\tau = (k/t)(\log(k/t))^{\beta\gamma'} + 2$, where $\gamma' = \min(\gamma/(\gamma + 1), \frac{1}{2})$, and

$$I(t) = \int_2^\tau + \int_\tau^\infty = I_1(t) + I_2(t), \quad \text{say.} \tag{14}$$

After the estimates in (1) we get

$$\begin{aligned} I_1(t) &\leqslant K_1 t \int_2^\tau (\log \omega)^\delta \, d\omega + K_2 t \int_2^\tau (\log \omega)^\beta \omega^{-1} e^{-\gamma}(\omega) m Q(\omega) \, d\omega \\ &\leqslant K_1 t \tau (\log \tau)^\delta + K_2 t \tau^{1-\gamma} (\log \tau)^{\delta + \beta\gamma} \\ &= O((\log(k/t))^\eta), \quad \text{uniformly in } t, 0 < t \leqslant \pi, \end{aligned} \tag{15}$$

since, for $0 < \gamma < 1$,

$$\begin{aligned} &\int_2^\tau (\log \omega)^\beta \omega^{-1} e^{-\gamma}(\omega) m Q(\omega) \, d\omega \\ &\qquad \leqslant \sum_{m < \tau} \int_m^{m+1} \{e(\omega) - e(m)\}^{\gamma - 1} \frac{(\log \omega)^\beta}{\omega e^\gamma(\omega)} e(m) \cdot (\log m)^\delta \, d\omega \\ &\qquad \leqslant K \sum_{m < \tau} (\log m)^\delta e^{-\gamma}(m + 1) \{e(m + 1) - e(m)\}^\gamma \\ &\qquad \leqslant K \sum_{m < \tau} m^{-\gamma} (\log m)^{\delta + \beta\gamma}. \end{aligned} \tag{16}$$

For $I_2(t)$, we first consider the case $0 < \gamma < 1$. Using the order estimates from (2)(i) and proceeding as for (15) and (16),

$$\begin{aligned} I_2(t) &\leqslant K_1 t^{-\gamma} \int_\tau^\infty (\log \omega)^{\beta(\gamma+1)} \omega^{-\gamma-1} \{(\log \omega)^{\delta - \beta} + (\log(\omega - k/t))^{\delta - \beta}\} \, d\omega \\ &\quad + K_2 \int_\tau^\infty (\log \omega)^\beta \omega^{-1} e^{-\gamma}(\omega) Q(\omega) \, d\omega \\ &\leqslant K_1 t^{-\gamma} \tau^{-\gamma} (\log \tau)^{\beta(\gamma+1)} \{(\log \tau)^{\delta - \beta} + (\log(\tau - k/t))^{\delta - \beta}\} \\ &\quad + K_2 \sum_{[\tau]}^\infty m^{-\gamma-1} (\log m)^{\delta + \beta\gamma} \\ &= O((\log(k/t))^\eta), \quad \text{uniformly in } t, 0 < t \leqslant \pi. \end{aligned} \tag{17}$$

For $\gamma \geqslant 1$, the order estimates in (2)(ii) give

$$I_2(t) \leqslant Kt^{-1}\int_{\tau}^{\infty}(\log \omega)^{\delta+\beta}\omega^{-2}\,d\omega$$
$$= O((\log(k/t))^{\eta}), \quad \text{uniformly in } t,\ 0 < t \leqslant \pi. \tag{18}$$

This completes the proof for Theorems II and III.

3.2. Proof of Theorem I. Note that $\delta < \eta$ implies that $\delta \leqslant \eta - \beta\gamma/(\gamma + 1)$, for all arbitrarily small positive γ, $0 < \gamma < 1$. We choose γ such that it is also less than α. Now by Theorem III, $\sum A_n(x)(\log n)^{\delta} \in |R, e(\omega), \gamma|$. Hence, by the First Theorem of Consistency for absolute Riesz summability (see [3, §1.9])

$$\sum A_n(x)(\log n)^{\delta} \in |R, e(\omega), \alpha|.$$

4. We obtain the following corollary to our Theorems I and II.

COROLLARY. *Let $\alpha > 0$ and δ and η be real numbers. If either*
(I) $\eta \neq 0$, $\delta < \eta$ *and* $\phi(t)(\log(k/t))^{\eta} \in BV(0, \pi)$, *or*
(II) $0 \leqslant \eta$, $\delta \leqslant \eta$ *and* $\int_0^{\pi}(\log(k/t))^{\eta}\,|d\phi(t)|$ *is finite*
then $\sum A_n(x)(\log n)^{\delta} \in |C, \alpha|$.

Part (I) of the corollary follows from Theorem I after a 'Second Theorem of Consistency' for absolute Riesz summability (see [4, Theorem 2]).

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