SOME FUNCTIONS WITH A UNIQUE INVARIANT MEAN

MICHEL TALAGRAND

Abstract. In a large class of groups, we construct a function which has a unique invariant mean, but which is not Riemann-measurable.

1. Introduction and notations. Let $G$ be a compact group. For $f \in L^\infty = L^\infty(G)$ and $a \in G$, let $f_a \in L^\infty$ be given by $f_a(x) = f(ax)$ for $x \in G$. An invariant mean on $L^\infty$ is a positive linear functional $m$ such that $m(1) = 1$ and $m(f_a) = m(f)$ for $f \in L^\infty$, $a \in G$. The normalized Haar measure $\lambda$ defines an invariant mean. We say $f \in L^\infty$ has a unique invariant mean if, for each invariant mean $m$, $m(f) = \int f \, d\lambda$. A sufficient condition for $f$ to have a unique invariant mean is the following:

1. For all $\varepsilon > 0$, there exist $a_1, \ldots, a_n$ in $G$ with $\|(1/n)\sum_{i=1}^n f_a - \int f \, d\lambda\| < \varepsilon$.

If $G$ is amenable as discrete, for example Abelian, it is known that this condition is also necessary.

It is clear that each continuous function has a unique invariant mean. An $f \in L^\infty$ will be said to be Riemann-measurable if it is (the class of) a function whose set of points of discontinuity is negligible. This is the same as saying that for all $\varepsilon > 0$, there exist continuous functions $g_1 < f < g_2$ with $\int (g_2 - g_1) \, d\lambda < \varepsilon$.

Then $f$ has a unique invariant mean. In fact, for each invariant mean $m$, we have $\int g_1 \, d\lambda < m(f) < \int g_2 \, d\lambda$ and both $\int g_1 \, d\lambda$ and $\int g_2 \, d\lambda$ differ from $\int f \, d\lambda$ by less than $\varepsilon$.

It is not hard to use (1) to construct in many groups functions $f$ which are not Riemann-measurable and have a unique invariant mean. A natural question, raised by L. A. Rubel and A. L. Shields [1, p. 39], is: Suppose for each continuous $g$, the function $fg$ has a unique invariant mean. Does it follow that $f$ is Riemann-measurable? We are going to give a negative answer to this question in a large class of groups.

2. Construction. The Haar measure of a measurable set $A \subset G$ will be denoted by $|A|$. If $A, B \subset G$, let $AB = \{ab: a \in A, b \in B\}$.

Theorem. Suppose $G$ satisfies the following condition:

2. $G$ is metrizable and there exist a sequence $(S_n)$ of measurable sets whose diameters go to zero and a sequence of finite sets $(P_n)$ of $G$ such that $S_1 = G$, and

Received by the editors November 20, 1979 and, in revised form, May 14, 1980.

1980 Mathematics Subject Classification. Primary 28C10; Secondary 26A15.

Key words and phrases. Unique invariant mean, Riemann-measurable.

This research was done while the author held a grant from NATO to visit Ohio State University.
such that, for $n > 1$, the sets $aS_{n+1}$ for $a \in P_{n+1}$ form a partition of $S_n$, and such that $Q_n = P_2 \cdot \cdot \cdot P_{n}$ is a finite subgroup of $G$.

Then there exists a function $f: G \to \{0, 1\}$, which is not Riemann-measurable, but such that $fh$ has a unique invariant mean whenever $h$ is Riemann-measurable.

In fact, by an obvious argument, it would be enough to suppose that $G$ has a quotient which satisfies (2). It is also not hard to see that (2) is satisfied whenever $G = \mathbb{R}/\mathbb{Z}$ or $G = (\mathbb{Z}/2\mathbb{Z})^\mathbb{N}$.

**Proof.** Let $Q_n = P_2P_3 \cdot \cdot \cdot P_n$. Hence the sets $aS_n$ for $a \in Q_n$ form a partition of $G$.

We shall construct by induction a sequence $X_k$ of subsets of $G$, and an increasing sequence of integers $n_k$ such that the following conditions hold:

(A) There exists $A \subset Q_n$ such that $X_k = AS_k$.

(B) For each $a \in Q_k$, $|X_k \cap aS_k| > |S_n|$, $|(G \setminus X_k) \cap aS_k| > |S_n|$.

(C) $|X_k \Delta X_{k-1}| < 2^{-k-2}|S_n|$ if $k > 2$.

(D) $\forall r \leq k \forall a \in Q_r, \sum_{b \in Q_k} \chi_{aS_n \cap b} = constant$.

Before we proceed to the construction, let us show why it implies the result. From the conditions (C), it follows that there exists a measurable set $X \subset G$ with

$$|X \Delta X_k| < \sum_{i \geq k} |X_i \Delta X_{i+1}| < |S_n| \sum_{i \geq k} 2^{-i-3} < \frac{1}{2}|S_n|.$$ 

The conditions (B) show that for $a \in Q_k$, $|X \cap aS_k| > 0$ and $|(G \setminus X) \cap aS_k| > 0$. Since the diameter of $S_k$ goes to zero, each open set contains a set $aS_k$ for $k$ large and $a \in Q_k$. Hence $X$ and its complement meet each open set in a set of positive measure. It shows that $f = x_X$ is not Riemann-measurable.

Now if in condition (D) we fix $r$ and let $k$ go to infinity, we get that for all $a \in Q_r$, we have

$$\sum_{b \in Q_n} \chi_{aS_n}(bx)f(bx) = constant$$

and hence from (1) the function $\chi_{aS_k}f$ has a unique invariant mean. Now, each continuous function $g$ can be uniformly approximated by a function of the type $\sum_{\alpha} a_\alpha \chi_{aS_k}$ for $p$ large; hence $fg$ has a unique invariant mean. Now, if $h$ is Riemann-measurable, for each $\epsilon > 0$ there exist continuous functions $g_1, g_2$ with $g_1 < h < g_2$, $\int (g_2 - g_1) \, d\lambda < \epsilon$, and hence, for each invariant mean $m$, $\int f g_1 \, d\lambda < m(fh) < \int f g_2 \, d\lambda$ where $\int f (g_2 - g_1) \, d\lambda < \epsilon$, which shows that $fh$ has a unique invariant mean.

We start the construction by picking $a \in P_2$, and setting $n_1 = 2$, $X_1 = aS_{n_1}$. Then (A), (B), (D) are satisfied and (C) is empty.

Now suppose the construction of $X_k$ has been done, such that the conditions (A) to (D) are satisfied. Enumerate $Q_{k+1}$ as $(a_q)_{1 \leq q \leq l}$ where $l = card Q_{k+1}$. The construction of $X_{k+1}$ will be done in $2l$ steps. We are going to construct for $p < 2l$ sets $Y_p$ of $G$ and increasing integers $m_p$, with $Y_1 = X_k$, $m_1 = n_k$, such that the following conditions hold.

(E) There exist $B \subset Q_{m_p}$ such that $Y_p = BS_{m_p}$. 

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(F_p) If \( p = 2q \), then \(|Y_p \cap a_q S_{k+1}| > |S_m|\). If \( p = 2q - 1 \) then \(|(G \setminus Y_p) \cap a_q S_{k+1}| > |S_m|\).

\((G_p)\) \( |Y_p\Delta Y_{p-1}| < 2^{-p-2-k}|S_{m_{p-1}}| \) if \( p > 2 \).

\((H_p)\) \( \forall r < k, \forall a \in Q_r, \sum_{b \in Q_a} x_{Y_p \cap aS}(bx) = \text{constant} \).

Before we perform the construction, let us show that \( X_{k+1} = Y_{2l} \) satisfies \((A_{k+1})\) to \((D_{k+1})\) if we take \( n_{k+1} = m_{2l} \). First, \((A_{k+1})\) is nothing else than \((E_{2l})\). Second, \((D_{k+1})\) is the union of \((H_{2l})\) and the following (which is clearly true)

\[ \forall a \in Q_{k+1}, \sum_{b \in Q_{n_{k+1}}} x_{Y_{2l} \cap aS_{k+1}}(bx) = \text{constant} \]

From the conditions \((G_p)\) we get that

\[ |X_k \Delta X_{k+1}| = |Y_1 \Delta Y_{2l}| < \sum_{p=2}^{2l} |Y_p \Delta Y_{p-1}| < \sum_{r \geq 2} 2^{-r-2-k}|S_{m_1}| < 2^{-k-3}|S_{m_1}| \]

and this is \((C_{k+1})\). By the same argument we have \(|Y_p \Delta X_{k+1}| < \frac{1}{2}|S_{m_1}| \) for all \( p \).

Now if \( a \in Q_{k+1} \), let \( q \) such that \( a = a_q \). From \((F_{2q})\) we get

\[ |X_{k+1} \cap aS_{k+1}| > |Y_{2q} \cap aS_{k+1}| - |Y_{2q} \Delta X_{k+1}| > \frac{1}{2}|S_{m_{2q}}| \]

and hence from \((A_{k+1}), |X_{k+1} \cap aS_{k+1}| > |S_{m_{k+1}}|\). We get similarly \(|(G \setminus X_{k+1}) \cap aS_{k+1}| > |S_{m_{k+1}}|\) by considering \( 2q - 1 \) instead of \( 2q \). Suppose now that (for example, the case \( 2q - 1 \) being similar) \( Y_{2q} \) and \( m_{2q} \) have been constructed satisfying \((E_{2q})\) to \((H_{2q})\). If \(|(G \setminus Y_{2q}) \cap a_{q+1} S_{k+1}| > 0 \) then this measure is greater than \(|S_{m_{2q}}|\) and it is enough to take \( Y_{2q+1} = Y_{2q}, m_{2q+1} = m_{2q} \). If not, let \( m_{2q+1} \) be an integer such that

\[ (3) 2^{2q+4+k} \text{ card } Q_k|S_{m_{2q+1}}| < |S_{m_{2q}}| \]

Since \(|Y_1 \Delta Y_{2q}| < \frac{1}{2}|S_{m_1}|\), it follows from \((B_k)\) that \(|(G \setminus Y_{2q}) \cap vS_k| > 0 \) for \( v \in Q_k \). Take for \( v \) the unique element in \( Q_k \) such that \( vS_k \subseteq a_{q+1} S_{k+1} \), i.e. such that \( a_{q+1} = vw \) for a certain \( w \in P_{k+1} \).

Set \( Q' = P_{k+1}P_{k+3} \cdots P_{m_{2q}}, Q'' = P_{m_{2q}+1}P_{m_{2q}+2} \cdots P_{m_{2q+1}}, \) and \( S = S_{m_{2q}}, S' = S_{m_{2q+1}} \). Since \(|(G \setminus Y_{2q}) \cap vS_k| > 0 \), it follows from \((E_{2q})\) that there exist \( w' \in P_{k+1} \) and \( x \in Q' \) with \( vw'xS \cap Y_{2q} = \emptyset \). Since we have assumed \( a_{q+1} S_{k+1} \subseteq Y_{2q} \), we have \( vw'xS \subseteq Y_{2q} \). Now let \( y \) be any fixed element of \( Q'' \). We have \( vw'xyS' \cap Y_{2q} = \emptyset \), \( vw'xyS' \subseteq Y_{2q} \). Each element \( u \in Q = Q_{m_{2q+1}} \) can be written in a unique way, \( u_1u_2u_3u_4 \), where \( u_1 \in Q_k, u_2 \in P_{k+1}, u_3 \in Q', \mu_4 \in Q'' \). Let \( T \) be the bijection of \( Q \) which lets \( u \) be fixed if \( u \notin \{ w, w' \} \) or \( u 
eq x \) or \( u 
eq y \) and which exchanges \( u_1w'xy \) and \( u_1wxy \) for each \( u_1 \).

From \((E_{2q})\) we have \( Y_{2q} = BS \) for \( B \subseteq Q_{m_{2q}} \) and hence \( Y_{2q} = CS' \) where \( C = BQ'' \). Define \( Y_{2q+1} = T(C)S' \). It is obvious that \((E_{2q+1})\) is satisfied. Since \( T \) moves at most \( 2 \text{ card } Q_k \) points of \( Q \), we have

\[ |Y_{2q} \Delta Y_{2q+1}| < 2 \text{ card } Q_k|S'| < 2^{-2q-3-k}|S| \]

from condition \((3)\); that is, \((G_{2q+1})\) holds and \((F_{2q+1})\) follows from the fact that, since \( vw'xyS' \cap Y_{2q} = \emptyset \), we have \( vwxyS' \cap Y_{2q+1} = \emptyset \) and \( vw = a_{q+1} \).
To complete the proof, it remains to show that \((H_{2q+1})\) holds. Fix \(r < k\) and \(a \in Q_r\), and define

\[
\alpha(x) = \sum_{b \in Q_k} \chi_{Y_{2q+1} \cap aS_k}(bx) = \sum_{b \in Q_k, t \in T(C)} \chi_{tS' \cap aS_k}(bx).
\]

We have \(S_r = RS'\), where \(R = P_{r+1}P_{r+2} \cdots P_{m_{2q+1}}\), so

\[
\alpha(x) = \sum \chi_{tS' \cap aS_k}(bx),
\]

where the summation is taken over \(b \in Q_k, t \in T(C), t' \in R\). We can write in a unique way \(x = b'z' x'\), where \(x' \in S', b' \in Q_k, z \in R' = P_{n+1}P_{n+2} \cdots P_{m_{2q+1}}\).

Since \(Q_{n}\) is a group, we have

\[
\alpha(x) = \sum \chi_{tS' \cap aS_k}(bzx')
\]

where the summation is taken over \(b \in Q_k, t \in T(C), t' \in R\). We have

\[
\chi_{tS' \cap aS_k}(bzx') = 1 \iff t = at' = bz.
\]

Since \(T\) is a bijection, and \(T^{-1} = T\), we have

\[
\chi_{tS' \cap aS_k}(bzx') = \chi_{T(t)S' \cap T(at)S}(T(bz)x').
\]

It is clear that \(T(at') = aT'(t')\) where \(T'\) is a bijection of \(R\). It is also easy to see that \(T(bz) = T''(b)z'\) where \(T''\) is a bijection of \(Q_{n}\) and \(z' \in R'\) is independent of \(b\). So we have

\[
\alpha(x) = \sum \chi_{tS' \cap aS_k}(b'z' x')
\]

where the summation is taken over \(t \in C, t' \in R, b \in Q_k\) so

\[
\alpha(x) = \sum_{t \in C} \chi_{tS' \cap aS_k}(b'z' x') = \sum_{b \in Q_k} \chi_{Y_{2q} \cap aS_k}(bz' x').
\]

From \((H_{2q})\) it follows that this quantity does not depend on \(z' x'\), hence of \(x\), which proves \((H_{2q+1})\) and concludes the proof.

**Acknowledgement.** The author thanks J. M. Rosenblatt for bringing this problem to his attention.

**References**


**Equipe d’Analyse-Tour 46, Université Paris VI, 4 Place Jussieu, 75230 Paris Cedex 05, France**