

## A COUNTABLE BASIS FOR $\Sigma_2^1$ SETS AND RECURSION THEORY ON $\aleph_1$

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**ABSTRACT.** Countably many  $\aleph_1$ -recursively enumerable sets are constructed from which all the  $\aleph_1$ -recursively enumerable sets can be generated by using countable union and countable intersection. This implies under  $V = L$  that there exists as well a countable basis for  $\Sigma_n^1$  sets of reals,  $n > 2$ . Further under  $V = L$  the lattice  $\mathcal{E}^*(\aleph_1)$  of  $\aleph_1$ -recursively enumerable sets modulo countable sets has only  $\aleph_1$  many automorphisms.

Let  $\mathcal{E}$  denote the lattice of recursively enumerable (r.e.) sets under inclusion, and let  $\mathcal{E}^*$  denote the quotient lattice of  $\mathcal{E}$  modulo the ideal of finite sets. Both structures have been extensively studied (see e.g. the survey by Soare [5]). In recent years research has concentrated on the existence of automorphisms and the decidability of the elementary theory.

Analogous questions arise in  $\alpha$ -recursion theory for admissible ordinals  $\alpha$ . Here one studies the lattice  $\mathcal{E}(\alpha)$  of  $\alpha$ -r.e. subsets of  $\alpha$  and the quotient lattice  $\mathcal{E}^*(\alpha)$  modulo the ideal of  $\alpha^*$ -finite sets (see e.g. the survey by Lerman [2]). A set is  $\alpha$ -r.e. iff it is definable over  $L_\alpha$  by some  $\Sigma_1$  formula with parameters. A function is  $\alpha$ -recursive iff its graph is  $\alpha$ -r.e. A set is  $\alpha^*$ -finite iff every  $\alpha$ -r.e. subset of it is  $\alpha$ -recursive. For simplicity we assume  $V = L$  in the first part of this paper where we study  $\aleph_1$ -r.e. sets.

Lachlan has proved the following basic result about automorphisms of  $\mathcal{E}^*$  (see Soare [4]): There are  $2^{\aleph_0}$  automorphisms of  $\mathcal{E}^*$ . Sutner [7] has noticed that one can use Lachlan's construction in order to show that for all countable admissible  $\alpha$  there are  $2^{\aleph_0}$  many automorphisms of  $\mathcal{E}^*(\alpha)$ . The argument breaks down for  $\alpha = \aleph_1$  despite the fact that  $\aleph_1$  is like  $\omega$  a regular cardinal. Observe that in the case  $\alpha = \aleph_1$  the  $\alpha^*$ -finite sets are just the countable sets. We show in this paper that there are in fact only  $\aleph_1$  (instead of  $2^{\aleph_1}$ ) many automorphisms of  $\mathcal{E}^*(\aleph_1)$ .

**DEFINITION 1.** We say that a class  $\Gamma$  of sets has a countable basis  $(A_n)_{n \in \omega}$  if  $\{A_n | n \in \omega\} \subseteq \Gamma$  and  $\Gamma$  is the closure of  $\{A_n | n \in \omega\}$  under countable unions and intersections.

Observe that the class of  $\aleph_1$ -r.e. sets is closed under countable unions and intersections.

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**THEOREM 2.** *The class of  $\aleph_1$ -r.e. sets has a countable basis  $(A_n)_{n \in \omega}$ . In fact every  $\aleph_1$ -r.e. set can be written as a countable intersection of countable unions of countable intersections of the sets  $(A_n)_{n \in \omega}$ .*

**PROOF.** Take a universal  $\aleph_1$ -r.e. set  $W$  such that  $(W_e)_{\omega < e < \aleph_1}$  is an enumeration of all  $\aleph_1$ -r.e. sets, where  $W_e := \{\delta | \langle e, \delta \rangle \in W\}$ . Further take an  $\aleph_1$ -recursive function  $C$  from  $\aleph_1$  into  $\mathcal{P}(\omega)$  such that  $\{C(e) | e \in \aleph_1\}$  is a family of almost disjoint sets (i.e. every  $C(e)$  is infinite and  $C(e) \cap C(e')$  is finite for  $e \neq e'$ , see e.g. Kunen [1]).

We construct first countably many  $\aleph_1$ -r.e. sets  $(A_n)_{n \in \omega}$  such that for every  $e \in \aleph_1$  with  $e > \omega$

$$W_e - e = \left( \bigcup_{j \in \omega} \left( \bigcap \{A_n | n \in C(e) \wedge n > j\} \right) \right) - e.$$

The sets  $(A_n)_{n \in \omega}$  are constructed simultaneously in  $\aleph_1$  many steps. At step  $\gamma$  we determine for every  $n$  on which fact it depends whether or not  $\gamma$  is enumerated in  $A_n$ .

We assign in an  $\aleph_1$ -recursive way to every  $\gamma \in \aleph_1$  a function  $p_\gamma \in L_{\aleph_1}$  which maps  $\omega$  one-one onto  $\gamma + 1$ . For  $e < \gamma$  one might consider  $p_\gamma^{-1}(e)$  as the priority of the equality  $W_e = \bigcup_{j \in \omega} (\bigcap \{A_n | n \in C(e) \wedge n > j\})$  at step  $\gamma$ . We change priorities at every step because it is important that the priority list is never longer than  $\omega$ .

*Step  $\gamma$  ( $\omega < \gamma < \aleph_1$ ).* For  $n \in C(p_\gamma(0))$  we determine that  $\gamma$  is put in  $A_n$  if and only if  $\gamma$  is enumerated in  $W_{p_\gamma(0)}$ . For  $j > 0$  and  $n \in (C(p_\gamma(j)) - \bigcup_{j' < j} C(p_\gamma(j')))$  we determine that  $\gamma$  is put in  $A_n$  if and only if  $\gamma$  is enumerated in  $W_{p_\gamma(j)}$ . For  $n \in \omega - \bigcup_{e < \gamma} C(e)$  it does not matter whether we put  $\gamma$  in  $A_n$  or not.

It is obvious from the construction that the sets  $A_n$  are  $\aleph_1$ -r.e. Further for  $\omega < e < \gamma$  we have

$$\begin{aligned} \gamma \in \bigcup_{j \in \omega} \left( \bigcap \{A_n | n \in C(e) \wedge n > j\} \right) \\ \Leftrightarrow \gamma \in \bigcap \left\{ A_n | n \in C(e) \wedge n > \max \left( C(e) \cap \left( \bigcup \{(C(e')) | p_\gamma(e') < p_\gamma(e)\} \right) \right) \right\} \\ \Leftrightarrow \gamma \in W_e. \end{aligned}$$

So far we cannot generate every set  $W_e$  with countable unions and intersections from the basis elements without making mistakes at countably many points. Therefore we add countably many further  $\aleph_1$ -r.e. sets to the constructed basis elements  $(A_n)_{n \in \omega}$  which enable us to correct these mistakes. Let  $f$  be an  $\aleph_1$ -recursive function which maps  $\aleph_1$  one-one into  $\mathcal{P}(\omega)$ . Define  $\aleph_1$ -recursive sets  $(R_n)_{n \in \omega}$  by  $\gamma \in R_n : \Leftrightarrow n \in f(\gamma)$ . We add then the sets  $(R_n)_{n \in \omega}$  and  $(\aleph_1 - R_n)_{n \in \omega}$  to the basis. For every  $\gamma \in \aleph_1$  we have

$$\{\gamma\} = \bigcap_{n \in f(\gamma)} R_n \cap \bigcap_{n \notin f(\gamma)} (\aleph_1 - R_n).$$

Thus we can write every countable set as a countable union of countable intersections and the complement of every countable set as a countable intersection of countable unions of basis elements. Therefore we can correct every mistake on countably many points.

**COROLLARY 3.** *There are  $\aleph_1$  automorphisms of  $\mathcal{E}^*(\aleph_1)$ .*

**PROOF.** It is obvious that one can construct  $\aleph_1$  many  $\aleph_1$ -recursive permutations of  $\aleph_1$  which induce different automorphisms of  $\mathcal{E}^*(\aleph_1)$ . On the other hand every automorphism  $\Phi$  of  $\mathcal{E}^*(\aleph_1)$  preserves countable unions and intersections. Therefore  $\Phi$  is completely determined by the values  $(\Phi(A_n^*))_{n \in \omega}$ , where  $(A_n)_{n \in \omega}$  is a basis for the  $\aleph_1$ -r.e. sets and  $(A_n^*)_{n \in \omega}$  are the corresponding equivalence classes in  $\mathcal{E}^*(\aleph_1)$ .

We now leave  $\alpha$ -recursion theory and the assumption  $V = L$  and turn to descriptive set theory in ZFC. It makes sense to ask whether the classes  $\Sigma_n^1$  and  $\Pi_n^1$  have a countable basis according to Definition 1 since these classes are closed under countable union and intersection. Obviously if  $\Sigma_n^1$  has a countable basis then the complements of the basis elements form a basis for  $\Pi_n^1$  and vice versa. Observe that  $\Delta_1^1$ , the class of Borel sets, has a countable basis. If one chooses suitable basis elements one can generate the Borel hierarchy without using complementation.

**COROLLARY 4.** *Assume  $n \geq 2$  and  ${}^\omega\omega \subseteq L[a]$  for some  $a \subseteq \omega$ . Then  $\Sigma_n^1$  has a countable basis.*

**PROOF.** It is well known that for every  $m \geq 1$  a subset of  ${}^\omega\omega$  is  $\Sigma_{m+1}^1$  iff it is  $\Sigma_m$  definable over HC. Under the assumption  ${}^\omega\omega \subseteq L[a]$  we have  $HC = HC^{L[a]} = L_{\aleph_1}[a]$ . Thus the  $\Sigma_2^1$  sets are just the sets which are  $\Sigma_1$  definable over  $L_{\aleph_1}[a]$  and for  $m \geq 2$  the  $\Sigma_{m+1}^1$  sets are just the sets which are  $\Sigma_1$  definable over  $\langle L_{\aleph_1}[a], \epsilon, P_m \rangle$  with a suitable mastercode  $P_m$ . Since one can define a map which maps  ${}^\omega\omega$  one-one onto  $\aleph_1$  by a  $\Delta_1$  definition over  $L_{\aleph_1}[a]$ , it does not matter whether one considers subsets of  ${}^\omega\omega$  or of  $\aleph_1$ . Further the construction of a countable basis in the proof of Theorem 2 works as well for  $L_{\aleph_1}[a]$  and  $\langle L_{\aleph_1}[a], \epsilon, P_m \rangle$  instead of  $L_{\aleph_1}$ .

**REMARK 5.** Richard Mansfield has shown [3] that any countably generated  $\sigma$ -algebra consisting entirely of Lebesgue measurable sets does not contain all  $\Sigma_1^1$  sets. Therefore  $\Sigma_n^1$  has no countable basis in the sense of Definition 1 if all  $\Sigma_n^1$  sets are measurable. This implies that  $\Sigma_1^1$  never has a countable basis. Further, Solovay's model of ZFC where all projective sets are measurable [6] supplies an example where no  $\Sigma_n^1$  has a countable basis.

In addition Mansfield has given a complete answer for  $\Sigma_2^1$ : If  $\Sigma_2^1$  has a countable basis then  ${}^\omega\omega \subseteq L[a]$  for some  $a \subseteq \omega$  (to appear).

#### REFERENCES

1. K. Kunen, *Combinatorics*, Handbook of Mathematical Logic (J. Barwise, Ed.), North-Holland, Amsterdam, 1977.
2. M. Lerman, *Lattices of  $\alpha$ -recursively enumerable sets*, Proc. Second Sympos. Generalized Recursion Theory (Oslo, 1977), North-Holland, Amsterdam, 1978.
3. R. Mansfield, *The solution to one of Ulam's problems concerning analytic sets. II*, Proc. Amer. Math. Soc. 26 (1970), 539–540.

4. R. I. Soare, *Automorphisms of the lattice of recursively enumerable sets. I, Maximal sets*, Ann. of Math. (2) **100** (1974), 80–120.
5. \_\_\_\_\_, *Recursively enumerable sets and degrees*, Bull. Amer. Math. Soc. **84** (1978), 1149–1181.
6. R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. (2) **92** (1970), 1–56.
7. K. Sutner, *Automorphisms of  $\alpha$ -recursively enumerable sets*, Diplomarbeit an der Universität München, 1979.

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