A PROOF OF THE BURKHOLDER THEOREM
FOR MARTINGALE TRANSFORMS

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Abstract. If \( g \) is the transform of an \( L^1 \)-bounded martingale \( f \) under a predictable sequence \( v \) satisfying \( \sup_n |v_n| < \infty \) almost everywhere, then a proof of the convergence of \( g \) is given using an approximation of \( f \) by a martingale of bounded variation.

Let \((\Omega, A, P)\) be a probability space, and \( M^1 \) the space of \( L^1 \)-bounded martingales \( f = (f_1, f_2, \ldots) \) relative to a fixed increasing sequence \( A_1, A_2, \ldots \) of sub-\( \sigma \)-fields of \( A \). Equipped with the norm \( \|f\|_1 = \sup_n \|f_n\|_1 \), \( M^1 \) is a Banach space.

A martingale \( f \), with \( f_n = \sum_{k=1}^n d_k, n > 1 \), \((d_k = f_k - f_{k-1}, d_1 = f_1)\) is of bounded variation if \( \sum_{k=1}^\infty |d_k(\omega)| < \infty \) for almost all \( \omega \).

Let \( BV = \{f \in M^1 : f \text{ is of bounded variation}\} \). Then, \( BV \) is dense in \( M^1 \) in \( M^1 \)-norm (Theorem 1 of [3, p. 166]).

The following basic convergence theorem is well known:

**Theorem (Theorem 1 of [1]).** Let \( f = (f_1, f_2, \ldots) \) be an \( L^1 \)-bounded martingale and let \( v = (v_1, v_2, \ldots) \) be a predictable sequence of random variables: \( v_k : \Omega \to \mathbb{R} \) is \( A_{k-1} \)-measurable, \( k > 1 \), such that \( \sup_n |v_n| < \infty \) a.e. Then the martingale transform \( g = (g_1, g_2, \ldots) \), defined by \( g_n = \sum_{k=1}^n v_k d_k \), converges a.e.

What is not so transparent is the mechanism of convergence for martingale transforms, i.e., Burkholder transforms. Here is a proof:

**Proof.** By a result of Burkholder and Shintani (Theorem 1 of [3]), for \( f \) in \( M^1 \) and arbitrary \( \epsilon > 0 \) there is a martingale \( f^{(\epsilon)} \) in \( BV \) such that \( \|f - f^{(\epsilon)}\|_1 < \epsilon^2 \). Let

\[
g_n^{(\epsilon)}(\omega) = \sum_{k=1}^n v_k d_k^{(\epsilon)}, \quad d_k^{(\epsilon)} = f_k^{(\epsilon)} - f_{k-1}^{(\epsilon)}, \quad k > 1.
\]

Then, for almost all \( \omega \in \Omega \),

\[
|g_n^{(\epsilon)}(\omega)| \leq \sum_{k=1}^n |v_k(\omega)| \cdot |d_k^{(\epsilon)}(\omega)| < \sup_n |v_n(\omega)| \cdot \sum_{k=1}^\infty |d_k^{(\epsilon)}(\omega)| < \infty.
\]

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This means that the sequence \( \{ g^{(e)}_{n}(\omega), n > 1 \} \) converges absolutely for almost all \( \omega \). So, \( P(\limsup_{m,n \to \infty} |g^{(e)}_m - g^{(e)}_n| > \varepsilon) = 0 \). Then

\[
P(\limsup_{m,n \to \infty} |g_m - g_n| > 3\varepsilon) = P(\limsup_{m,n \to \infty} (|g_m - g^{(e)}_m| + |g^{(e)}_n - g_n| + |g^{(e)}_m - g^{(e)}_n|) > 3\varepsilon)
\]

\[
< P(\limsup_{m,n \to \infty} |g_m - g^{(e)}_m| > \varepsilon) + P(\limsup_{m,n \to \infty} |g^{(e)}_n - g_n| > \varepsilon) + P(\limsup_{m,n \to \infty} |g^{(e)}_m - g^{(e)}_n| > \varepsilon)
\]

\[
= 2 \cdot P(\inf_{m > 1} (\sup_{m < n} |g_n - g^{(e)}_n|) > \varepsilon)
\]

\[
< 2 \cdot P(\sup_n |g_n - g^{(e)}_n| > \varepsilon).
\]

Now, by the weak \( L^1 \)-inequality of Burkholder, for each constant \( c > 0 \) there is a universal constant \( C > 0 \) such that if \( |v| < c \) uniformly then

\[
P(\sup_n |g_n| > \lambda) < C \cdot \lambda^{-1} \cdot \|f\|_1
\]

for \( f \in M^1 \) and all \( \lambda > 0 \). For a proof, see [2].

Therefore

\[
P(\limsup_{m,n \to \infty} |g_m - g_n| > 3\varepsilon) < 2C \cdot \varepsilon^{-1} \cdot \|f - f^{(e)}\|_1
\]

\[
< 2C \cdot \varepsilon \quad \text{for all } \varepsilon > 0.
\]

Since \( \sup_n |v_n| < \infty \) a.e., this means that \( \{ g_n(\omega), n > 1 \} \) is a Cauchy sequence for almost all \( \omega \). Since the state space \( X = \mathbb{R} \) is complete, \( \lim_{n \to \infty} g_n(\omega) \) exists for almost all \( \omega \) and belongs to \( X \). This implies that \( g \) converges a.e. and the theorem is proved.

REFERENCES


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