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REGULARITY CONDITIONS AND INTERSECTING HYPERGRAPHS

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ABSTRACT. Let $(\mathcal{F}, X)$ be a hypergraph with a transitive group of automorphisms. Suppose further that any four edges of $\mathcal{F}$ intersect nontrivially. Denoting $|X|$ by n we prove $|\mathcal{F}| = O(2^n)$. We show as well that it is not sufficient to suppose regularity instead of the transitivity of $\mathrm{Aut}(\mathcal{F})$.

1. Introduction. Let $(\mathcal{F}, X)$ be a hypergraph, i.e. $\mathcal{F}$ is a family of nonempty subsets of X. Let $|X| = n$.

We say $\mathcal{F}$ is k-intersecting if any k edges of $\mathcal{F}$ have a nonempty intersection. Obviously a k-intersecting hypergraph is k'-intersecting for every $k' < k$.

Erdös, Ko and Rado [2] observed that a 2-intersecting hypergraph has at most 2^{n-1} edges, moreover if it is 3-intersecting, then $|\mathcal{F}| = 2^{n-1}$ is possible only if $\mathcal{F}$ consists of all the subsets of X containing a fixed element of X.

Brace and Daykin [1] refined this result by proving that if there is no element of X which is contained in every edge of the k-intersecting hypergraph $\mathcal{F}$, then

$$|\mathcal{F}| < (k + 2)2^{n-k-1}. \tag{1}$$

To obtain equality in (1) we have to take all the subsets of X which contain at least k elements of a fixed $(k + 1)$-subset of X.

For $k = 2$, (1) gives 2^{n-1}. If n is odd then the family of subsets of X with cardinality exceeding $n/2$ gives a 2-intersecting hypergraph with transitive group of automorphisms and cardinality 2^{n-1}. For n even the family still has cardinality $2^{n-1}(1 + o(1))$.

In §2 we prove the following

THEOREM 1. *Suppose* $\mathrm{Aut}(\mathcal{F})$ *is transitive on* X, *and* $\mathcal{F}$ *is* 4-*intersecting. Then*

$$|\mathcal{F}| = o(2^n). \tag{2}$$

For the proof we need the following result of [3]:

THEOREM 2. *Suppose any three edges of* $(\mathcal{F}, X)$ *have at least* t *elements in common. Let us set* $b = (\sqrt{5} - 1)/2$. *Then we have*

$$|\mathcal{F}| < b^t 2^n. \tag{3}$$

We say $(\mathcal{F}, X)$ is regular if every element of X is contained in the same number of edges. In §3 we construct k-intersecting regular hypergraphs with

$$|\mathcal{F}| = 2^{n-(2^{k+1}-k-2)}.$$

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The construction uses the k-dimensional projective space over $GF(2)$.

2. The proof of Theorem 1. Let us set

$$t = \min\{|F_1 \cap F_2 \cap F_3|: F_1, F_2, F_3 \in \mathfrak{F}\}.$$

We consider two cases.

Case (a). $t \geqslant \log n - 3 \log \log n$. (log means $\log_2$.) By Theorem 2,

$$|\mathfrak{F}| \leqslant b^{(\log n - 3 \log \log n)} 2^n. \tag{4}$$

Now (2) follows from (4). For $n > n_0$, (4) implies that $|\mathfrak{F}| < 2^n/\sqrt{n}$.

Case (b).

$$t < \log n - 3 \log \log n. \tag{5}$$

Let F, G, H be members of $\mathfrak{F}$ satisfying $|F \cap G \cap H| = t$. Let us set

$$\mathfrak{D} = \{a(F) \cap a(G) \cap a(H) \mid a \in \operatorname{Aut}(\mathfrak{F})\}.$$

(By $a(F)$ we denote the elementwise image of F by the automorphism a.) By the definition of $\mathfrak{D}$ we have $\operatorname{Aut}(\mathfrak{F}) \subseteq \operatorname{Aut}(\mathfrak{D})$, in particular every element of X is contained in the same number, say d, of members of $\mathfrak{D}$. By an elementary count

$$d = t|\mathfrak{D}|/n. \tag{6}$$

Let us choose pairwise disjoint members $D_1, \ldots, D_m$ of $\mathfrak{D}$ such that for every member D_{m+1} of $\mathfrak{D}$ at least one of the intersections $D_i \cap D_{m+1}$ $(i = 1, \ldots, m)$ is nonempty. As $S = D_1 \cup \cdots \cup D_m$ has cardinality mt and has a nonempty intersection with every member of $\mathfrak{D}$, some vertex of S is contained in at least $|\mathfrak{D}|/mt$ members of $\mathfrak{D}$. Taking (6) into account we obtain

$$m \geqslant n/t^2. \tag{7}$$

We assert that for $i = 1, \ldots,$ and any $F \in \mathfrak{F}$ we have $D_i \cap F \neq \emptyset$. Indeed by the definition of $\mathfrak{D}$ we can find F_1, F_2, F_3 such that $D_i = F_1 \cap F_2 \cap F_3$, and using the 4-intersection property of $\mathfrak{F}$ we deduce

$$D_i \cap F = F_1 \cap F_2 \cap F_3 \cap F \neq \emptyset.$$

Now the number of subsets of X which have a nonempty intersection with every D_i, $i = 1, \ldots, m$, gives an upper bound for $|\mathfrak{F}|$, that is

$$|\mathfrak{F}| \leqslant 2^{n-mt}(2^t - 1)^m. \tag{8}$$

From (8) using (5) and (7) we deduce

$$|\mathfrak{F}| \leqslant 2^n\left(1 - \frac{1}{2^t}\right)^{n/t^2} < 2^n \exp(-\log_2 n) < 2^n/n. \tag{9}$$

Now the statement of the theorem follows from (9). Let us close this paragraph with a conjecture.

Conjecture 1. If in Theorem 1 we replace "4-intersecting" by "3-intersecting", then (2) remains valid.

3. Construction of k-intersecting regular hypergraphs. Let $(\mathcal{G}, Y)$ be the hypergraph consisting of the $(k-1)$-dimensional subspaces of the k-dimensional projective space over $GF(2)$. Then $|Y| = 2^{k+1} - 1$, $|\mathcal{G}| = |Y|$, and every element of Y is contained in exactly $2^k - 1$ of the members of $\mathcal{G}$, which all have cardinality

$2^k - 1$. Let y be an arbitrary element of Y, and let us define $Z = Y - \{y\}$, $\mathcal{R} = \{P \in \mathcal{P} \mid y \notin P\}$. Then for the hypergraph $(\mathcal{R}, Z)$ we have

$$|Z| = 2^{k+1} - 2, \qquad |\mathcal{R}| = (2^{k+1} - 1) - (2^k - 1) = 2^k,$$
$$|R| = 2^k - 1 \quad \text{for every } R \in \mathcal{R}.$$

Moreover, as the group of automorphisms of $(\mathcal{P}, Y)$ is doubly transitive on Y, $\mathrm{Aut}(\mathcal{R})$ is transitive on Z. Hence the hypergraph $(\mathcal{R}, Y)$ is regular. Consequently every point of Z is contained in $|\mathcal{R}|\,|R|/|Z| = \frac{1}{2}|\mathcal{R}| = 2^{k-1}$ members of $\mathcal{R}$. For our purposes the most important property of $\mathcal{P}$, and so of $\mathcal{R}$, is that any k members of it have a nonempty intersection.

Let X be an n-element set containing Z, and let us define

$$\mathcal{F} = \{F \subset X \mid (F \cap Z) \in \mathcal{R}\}.$$

As $\mathcal{R}$ is k-intersecting, $\mathcal{F}$ is k-intersecting as well. Using the definition of $\mathcal{F}$ we obtain

$$|\mathcal{F}| = |\mathcal{R}| 2^{n-|Z|} = 2^k 2^{n-(2^{k+1}-2)} = 2^{n-(2^{k+1}-k-2)};$$
$$|\{F \in \mathcal{F} \mid z \in F\}| = \left(\tfrac{1}{2}|\mathcal{R}|\right)(2^{n-|Z|}) = \tfrac{1}{2}|\mathcal{F}| \qquad (z \in Z);$$
$$|\{F \in \mathcal{F} \mid x \in F\}| = (|\mathcal{R}|)\left(\tfrac{1}{2}2^{n-|Z|}\right) = \tfrac{1}{2}|\mathcal{F}| \qquad (x \in (X - Z)).$$

So we have constructed a regular, k-intersecting hypergraph on n vertices with $2^{n-(2^{k+1}-k-2)}$ edges. Could we have done better?

Conjecture 2. A regular, k-intersecting hypergraph on n vertices has at most $2^{n-(2^{k+1}-k-2)}$ edges when $k \geqslant 3$.

The best upper bound we can prove for the moment is $2^n b^{2^{k-3}}$.

References

1. A. Brace and D. E. Daykin, *Sperner type theorems for finite sets*, Bull. Austral. Math. Soc. **5** (1971), 197–202.

2. P. Erdős, C. Ko and R. Rado, *Intersection theorems for finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320.

3. P. Frankl, *Families of finite sets satisfying an intersection condition*, Bull. Austral. Math. Soc. **15** (1976), 73–79.

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