REGULARITY CONDITIONS AND INTERSECTING HYPERGRAPHS

PETER FRANKL

Abstract. Let \((\mathcal{F}, X)\) be a hypergraph with a transitive group of automorphisms. Suppose further that any four edges of \(\mathcal{F}\) intersect nontrivially. Denoting \(|X|\) by \(n\) we prove \(|\mathcal{F}| = O(2^n)\). We show as well that it is not sufficient to suppose regularity instead of the transitivity of \(\text{Aut}(\mathcal{F})\).

1. Introduction. Let \((\mathcal{F}, X)\) be a hypergraph, i.e. \(\mathcal{F}\) is a family of nonempty subsets of \(X\). Let \(|X| = n\).

We say \(\mathcal{F}\) is \(k\)-intersecting if any \(k\) edges of \(\mathcal{F}\) have a nonempty intersection. Obviously a \(k\)-intersecting hypergraph is \(k'\)-intersecting for every \(k' < k\).

Erdős, Ko and Rado [2] observed that a 2-intersecting hypergraph has at most \(2n - 1\) edges, moreover if it is 3-intersecting, then \(|\mathcal{F}| = 2^n - 1\) is possible only if \(\mathcal{F}\) consists of all the subsets of \(X\) containing a fixed element of \(X\).

Brace and Daykin [1] refined this result by proving that if there is no element of \(X\) which is contained in every edge of the \(k\)-intersecting hypergraph \(\mathcal{F}\), then

\[|\mathcal{F}| < (k + 2)2^{n-k-1}.\]  

To obtain equality in (1) we have to take all the subsets of \(X\) which contain at least \(k\) elements of a fixed \((k + 1)\)-subset of \(X\).

For \(k = 2\), (1) gives \(2^n - 1\). If \(n\) is odd then the family of subsets of \(X\) with cardinality exceeding \(n/2\) gives a 2-intersecting hypergraph with transitive group of automorphisms and cardinality \(2^n - 1\). For \(n\) even the family still has cardinality \(2^n - 1(1 + o(1))\).

In §2 we prove the following

Theorem 1. Suppose \(\text{Aut}(\mathcal{F})\) is transitive on \(X\), and \(\mathcal{F}\) is 4-intersecting. Then

\[|\mathcal{F}| = o(2^n).\]  

For the proof we need the following result of [3]:

Theorem 2. Suppose any three edges of \((\mathcal{F}, X)\) have at least \(t\) elements in common. Let us set \(b = (\sqrt[3]{5} - 1)/2\). Then we have

\[|\mathcal{F}| < b' 2^n.\]  

We say \((\mathcal{F}, X)\) is regular if every element of \(X\) is contained in the same number of edges. In §3 we construct \(k\)-intersecting regular hypergraphs with

\[|\mathcal{F}| = 2^n - (2^k - 1 - k - 2).\]
The construction uses the \( k \)-dimensional projective space over \( GF(2) \).

2. The proof of Theorem 1. Let us set

\[
t = \min \{ |F_1 \cap F_2 \cap F_3| : F_1, F_2, F_3 \in \mathcal{F} \}.
\]

We consider two cases.

*Case (a).* \( t > \log n - 3 \log \log n \) (log means \( \log_2 \)). By Theorem 2,

\[
|\mathcal{F}| < \exp(\log n - 3 \log \log n)2^n. \tag{4}
\]

Now (2) follows from (4). For \( n > n_0 \), (4) implies that \( |\mathcal{F}| < 2^n/\sqrt{n} \).

*Case (b).*

\[
t < \log n - 3 \log \log n. \tag{5}
\]

Let \( F, G, H \) be members of \( \mathcal{F} \) satisfying \( |F \cap G \cap H| = t \). Let us set

\[
\mathcal{D} = \{ a(F) \cap a(G) \cap a(H) \mid a \in \text{Aut}(\mathcal{F}) \}.
\]

(By \( a(F) \) we denote the elementwise image of \( F \) by the automorphism \( a \).) By the definition of \( \mathcal{D} \) we have \( \text{Aut}(\mathcal{F}) \subseteq \text{Aut}(\mathcal{D}) \), in particular every element of \( X \) is contained in the same number, say \( d \), of members of \( \mathcal{D} \). By an elementary count

\[
d = t|\mathcal{D}|/n. \tag{6}
\]

Let us choose pairwise disjoint members \( D_1, \ldots, D_m \) of \( \mathcal{D} \) such that for every member \( D_{m+1} \) of \( \mathcal{D} \) at least one of the intersections \( D_i \cap D_{m+1} \) \( (i = 1, \ldots, m) \) is nonempty. As \( S = D_1 \cup \cdots \cup D_m \) has cardinality \( mt \) and has a nonempty intersection with every member of \( \mathcal{D} \), some vertex of \( S \) is contained in at least \( |\mathcal{D}|/mt \) members of \( \mathcal{D} \). Taking (6) into account we obtain

\[
m > n/t^2. \tag{7}
\]

We assert that for \( i = 1, \ldots, m \), and any \( F \in \mathcal{F} \) we have \( D_i \cap F \neq \emptyset \). Indeed by the definition of \( \mathcal{D} \) we can find \( F_1, F_2, F_3 \) such that \( D_i = F_1 \cap F_2 \cap F_3 \), and using the 4-intersection property of \( \mathcal{F} \) we deduce

\[
D_i \cap F = F_1 \cap F_2 \cap F_3 \cap F \neq \emptyset.
\]

Now the number of subsets of \( X \) which have a nonempty intersection with every \( D_i, i = 1, \ldots, m \), gives an upper bound for \( |\mathcal{F}| \), that is

\[
|\mathcal{F}| < 2^{n-mt}(2^t - 1)^m. \tag{8}
\]

From (8) using (5) and (7) we deduce

\[
|\mathcal{F}| < 2^n \left( 1 - \frac{1}{2^t} \right)^{n/t^2} < 2^n \exp(-\log_2 n) < 2^n/n. \tag{9}
\]

Now the statement of the theorem follows from (9). Let us close this paragraph with a conjecture.

*Conjecture 1.* If in Theorem 1 we replace “4-intersecting” by “3-intersecting”, then (2) remains valid.

3. Construction of \( k \)-intersecting regular hypergraphs. Let \( (\mathcal{F}, Y) \) be the hypergraph consisting of the \( (k-1) \)-dimensional subspaces of the \( k \)-dimensional projective space over \( GF(2) \). Then \( |Y| = 2^{k+1} - 1 \), \( |\mathcal{F}| = |Y| \), and every element of \( Y \) is contained in exactly \( 2^k - 1 \) of the members of \( \mathcal{F} \), which all have cardinality
$2^k - 1$. Let $y$ be an arbitrary element of $Y$, and let us define $Z = Y - \{y\}$, $\mathcal{R} = \{P \in \mathcal{P} \mid y \notin P\}$. Then for the hypergraph $(\mathcal{R}, Z)$ we have

$$|Z| = 2^{k+1} - 2, \quad |\mathcal{R}| = (2^{k+1} - 1) - (2^k - 1) = 2^k,$$

$$|R| = 2^k - 1 \quad \text{for every } R \in \mathcal{R}.$$  

Moreover, as the group of automorphisms of $(\mathcal{P}, Y)$ is doubly transitive on $Y$, $\text{Aut}(\mathcal{R})$ is transitive on $Z$. Hence the hypergraph $(\mathcal{R}, Y)$ is regular. Consequently every point of $Z$ is contained in $|\mathcal{R}| |R|/|Z| = \frac{1}{2} |\mathcal{R}| = 2^{k-1}$ members of $\mathcal{R}$. For our purposes the most important property of $\mathcal{P}$, and so of $\mathcal{R}$, is that any $k$ members of it have a nonempty intersection.

Let $X$ be an $n$-element set containing $Z$, and let us define

$$\mathcal{F} = \{F \subset X \mid (F \cap Z) \in \mathcal{R}\}.$$  

As $\mathcal{R}$ is $k$-intersecting, $\mathcal{F}$ is $k$-intersecting as well. Using the definition of $\mathcal{F}$ we obtain

$$|\mathcal{F}| = |\mathcal{R}| 2^n - |Z| = 2^k 2^n - (2^{k+1} - 2) = 2^n - (2^{k+1} - k - 2);$$

$$|\{F \in \mathcal{F} \mid z \in F\}| = \left(\frac{1}{2} |\mathcal{R}|\right) (2^n - |Z|) = \frac{1}{2} |\mathcal{F}| \quad (z \in Z);$$

$$|\{F \in \mathcal{F} \mid x \in F\}| = \left(\frac{1}{2} |\mathcal{R}|\right) (\frac{1}{2} 2^n - |Z|) = \frac{1}{2} |\mathcal{F}| \quad (x \in (X - Z)).$$

So we have constructed a regular, $k$-intersecting hypergraph on $n$ vertices with $2^n - (2^{k+1} - k - 2)$ edges. Could we have done better?

**Conjecture 2.** A regular, $k$-intersecting hypergraph on $n$ vertices has at most $2^n - (2^{k+1} - k - 2)$ edges when $k > 3$.

The best upper bound we can prove for the moment is $2^n b 2^{k-3}$.

**References**


**Centre National de la Recherche Scientifique, 15 Quai Anatole France, 75007 Paris, France**