PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 3, July 1981

MAXIMAL SEPARABLE INTERMEDIATE FIELDS OF LARGE CODEGREE

NICKOLAS HEEREMA

ABSTRACT. Let k be a function field in n ($n > 0$) variables over k_0, a field having characteristic $p \neq 0$. An intermediate field s is maximal separable if s/k_0 is separable and s is not properly contained in any subfield of k separable over k_0. The following result is proved. If $n = 1$ the set $\Delta = \{[k : s] \mid s \text{ maximal separable}\}$ is bounded if and only if the algebraic closure $\bar{k}_0$ of k_0 in k is separable over k_0. If $n > 1$ and Δ is bounded then $\bar{k}_0/k_0$ is separable. An upper bound for Δ is obtained for the case $n = 1$ and $\bar{k}_0/k_0$ separable.

I. Introduction. In 1947 Dieudonné initiated the study of maximal separable intermediate fields [2]. Let $k \supset k_0$ be fields of characteristic $p \neq 0$ with k finitely generated over k_0. Let r be the exponent of inseparability of k/k_0, that is, r is the least positive integer such that $k_0(k^{p^r})/k_0$ is separable. Dieudonné showed that the maximal separable intermediate fields s with the property $k \subset k_0^{p^{-\infty}}(s)$, which he called distinguished subfields of k/k_0, are those of the form $k_0(k^{p^r})(T)$ where T^{p^r} is a p-basis for $k_0(k^{p^r})/k_0$. He also observed that all such fields have minimal codegree $[k : k_0]_i$ among intermediate fields separable over k_0. In 1970 Kraft [5] showed that the distinguished subfields of k/k_0 are precisely those intermediate fields s separable over k_0 for which $[k : s]$ is minimal. Dieudonné showed by example that not all maximal separable intermediate fields are distinguished. Recently Deveney and Mordeson have reported results on the characterization of those field extensions k/k_0 for which every maximal separable intermediate field is distinguished. In this note we consider a complementary question. When is $\Delta = \{[k : s] \mid s \text{ a maximal separable subfield of } k/k_0\}$ bounded? Theorem 1 states that if the transcendency degree of k/k_0 is one then Δ is bounded if and only if the algebraic closure $\bar{k}_0$ of k_0 in k is separable over k_0. Corollary 1 provides an upper bound for Δ if $\bar{k}_0/k_0$ is separable and Corollary 2 asserts that, dropping the restriction on transcendency degree, if Δ is bounded then $\bar{k}_0/k_0$ is separable. We conjecture the converse having been unable to settle the matter.

The fields $k_0(k^{(n)}) = \{a \in k \mid \text{for some } t > 0,\ a^{p^t} \in k_0(k^{p^{t+n}})\}$ have an essential role in this paper. Their basic properties are investigated in [3].

II. Maximal separable fields. Throughout $k \supset k_0$ are fields of characteristic $p \neq 0$ with k finitely generated over k_0. The exponent of inseparability of k/k_0 is r. The abbreviation tr.deg.(k/k_0) denotes the transcendency degree of k/k_0. We begin with a statement (A) of basic and familiar facts.

Received by the editors June 11, 1980.
1980 *Mathematics Subject Classification.* Primary 12F15, 12F20.

(A) If k/k_0 is separable then (i) k/k_0 is separably generated and (ii) a p-basis for k/k_0 is a separating transcendency basis for k/k_0 and conversely.

The definition of a separable extension, namely s/k_0 is separable if s and $k_0^{p^{-1}}$ are linearly disjoint over k_0, implies that if U is a linear basis for s/k_0 then U^{p^n} is a linear basis for $k_0(s^{p^n})/k_0$ from which fact one derives the following.

(B) If $a^{p^r} \in k_0(k^{p^{n+r}}) \setminus k_0(k^{p^{n+r+1}})$ then $a \in k_0(k^{(n)}) \setminus k_0(k^{n+1})$ and conversely.

We shall also use the following result.

(C) [4, Lemma 2.4] An intermediate field s such that s/k_0 is separable and k/s is radical is a maximal separable intermediate field if and only if $k^p \cap s \subseteq k_0(s^p)$.

PROPOSITION 1. *If* tr.deg.$(k/k_0) = 1$ *and s is an intermediate field separable over k_0 with k/s a radical extension then, for some integer q, $q \geqslant 0$,* $k_0(s^{p^r}) = k_0(k^{p^{r+q}})$. *Also, the following are equivalent*:

(i) $k_0(s^{p^r}) = k_0(k^{p^{r+q}})$.

(ii) $s \subset k_0(k^{(q)}) \setminus k_0(k^{(q+1)})$.

(iii) $[k : s] = p^q[k : k_0]_i$.

PROOF. Since k/s is finitely generated radical $k_0(k^{p^m}) \subset s$ for some $m \geqslant 0$. Thus $k_0(k^{p^{m+r}}) \subset k_0(s^{p^r}) \subset k_0(k^{p^r})$. Since tr.deg.$(k/k_0) = 1$ and $k_0(k^{p^r})/k_0$ is separable $k_0(k^{p^r})$ has a one element p-basis over k_0 and thus $k_0(k^{p^r})$ is a simple extension of $k_0(k^{p^{r+m}})$. It follows that the only intermediate fields are those of the form $k_0(k^{p^{q+r}})$, $0 \leqslant q \leqslant m$, and so $k_0(s^{p^r}) = k_0(k^{p^{q+r}})$ for some q.

Assume (i). By statement (B), $s \subset k_0(k^{(q)}) \setminus k_0(k^{(q+1)})$. Conversely, if $s \subset k_0(k^{(q)}) \setminus k_0(k^{(q+1)})$ then, by (B), $k_0(s^{p^r}) \subset k_0(k^{p^{r+q}}) \setminus k_0(k^{p^{r+q+1}})$. Hence $k_0(s^{p^r}) = k_0(k^{p^{r+q}})$.

To show that (i) implies (iii) let s_1 be a distinguished subfield of k/k_0. Then $k_0(s_1^{p^r}) = k_0(k^{p^r})$ [5, Satz III, p. 113] and, assuming (i),

$$\begin{aligned}\left[k : k_0(k^{p^{r+q}})\right] &= \left[k : s_1\right]\left[s_1 : k_0(s_1^{p^r})\right]\left[k_0(s_1^{p^r}) : k_0(k^{p^{r+q}})\right] \\ &= \left[k : s\right]\left[s : k_0(s^{p^r})\right]. \qquad (1)\end{aligned}$$

By statement (A), p-bases for s and s_1 over k_0 have the same cardinality, namely one, and hence $[s_1 : k_0(s_1^{p^r})] = [s : k_0(s^{p^r})]$. Thus $[k : s] = [k : k_0]_i p^q$. Conversely, if (iii) then, by (1), $[k : k_0(k^{p^{r+q}})] = [k : k_0(s^{p^r})]$ and since $k_0(s^{p^r}) = k_0(k^{p^{r+m}})$ for some m it must be that $k_0(s^{p^r}) = k_0(k^{p^{r+q}})$.

PROPOSITION 2. *Assume that* tr.deg.$(k/k_0) = 1$. *If $k_0(k^{(q)}) \not\subset k_0(k^p)$ there is a maximal separable intermediate field s with the property $s \subset k_0(k^{(q)}) \setminus k_0(k^{(q+1)})$.*

PROOF. Let a be in $k_0(k^{(q)}) \setminus k_0(k^p)$. If a is in $k_0(k^{(q+1)})$ replace a by $a + b^{p^q}$ where $b \in s_1 \setminus k_0(s_1^p)$ for a distinguished subfield s_1 of k/k_0. Then by statement (B), the fact that $b^{p^r} \in k_0(s_1^{p^r}) \setminus k_0(s_1^{p^{r+1}})$ and noting that $k_0(s_1^{p^r}) = k_0(k^{p^r})$ we have $b^{p^q} \in k_0(k^{(q)}) \setminus k_0(k^{(q+1)})$. Thus we can assume that $a \in k_0(k^{(q)}) \setminus k_0(k^{(q+1)})$ and $a \notin k_0(k^p)$. Let $s = k_0(k^{p^{r+q}})(a)$. Since $a^{p^r} \in k_0(k^{p^{r+q}}) \setminus k_0(k^{p^{r+q+1}})$, a^{p^r} is a separating transcendency basis for $k_0(k^{p^{r+q}})/k_0$. It follows that s/k_0 is separable. Clearly k/s is radical. If s is not maximal separable then, by statement (C), $k^p \cap s \not\subset k_0(s^p)$. Hence there is an element b in k such that $b^p \in s \setminus k_0(s^p)$. Then

$\{b^p\}$ is a p-basis for s/k_0 and $s = k_0(s^p)(b^p) \subset k_0(k^p)$ contradicting the fact that a is in s. Thus s is maximal separable and, by construction, $s \subset k_0(k^{(q)})$.

PROPOSITION 3. *Assume* tr.deg.$(k/k_0) = 1$. *Let s be a maximal separable subfield of $k_0(k^{(q)})$ contained in $k_0(k^{(q+t)})$. If $s \subset s_1$ and s_1 is a maximal separable subfield of k/k_0 then $[k : s_1] \geqslant p^t[k : k_0]_i$.*

PROOF. Let $k_1 = k_0(k^{(q)})$. Then $k_0(k^{(q+t)}) = k_0(k_1^{(t)})$ [3, Lemma 4, p. 289] and so, by Proposition 1, $[k_1 : s] \geqslant p^t[k_1 : k_0]_i$. Since $s_1^{p^q} \subset k_1$ it follows that $k_0(s_1^{p^n}) \subset s$ for some n. Hence $s_1 = s(a)$ for some a and a^{p^q} is in k_1. Since $s(a^{p^q})$ is separable over k_0 and s is maximal separable in k_1 it follows that a^{p^q} is in s. Thus $[s_1 : s] \leqslant p^q$ and $[k : s_1] = [k : k_1][k_1 : s]/[s_1 : s] \geqslant [k : k_1][k_1 : k_0]_i p^{t-q}$. If s_2 is a distinguished subfield of k/k_0 then $k_0(s_2^{p^q})$ is a distinguished subfield of k_1 [3, Theorem 2, p. 288]. Thus, $[k : k_0]_i[s_2 : k_0(s_2^{p^q})] = [k : k_1][k_1 : k_0]_i$ or $[k : k_0]_i p^q = [k_1 : k_0]_i[k : k_1]$. Substituting this in the above inequality yields $[k : s_1] \geqslant [k : k_0]_i p^t$.

THEOREM 1. *If* tr.deg.$(k/k_0) = 1$ *the set $\Delta = \{[k : s] | s$ a maximal separable subfield of $k/k_0\}$ is bounded if and only if the algebraic closure of k_0 in k is a separable extension of k_0.*

PROOF. Let $\bar{k}_0$ be the algebraic closure of k_0 in k. Let $k_1 = k_0(k^{(q)})$ where q is chosen large enough so that $k_0(k^{(q)}) = \bar{k}_0(k^{p^q})$ and $k_0(k^{p^q})/k_0$ is separable [3, Theorem 9, p. 290]. If $\bar{k}_0/k_0$ is inseparable $\bar{k}_0$ has a nonempty p-basis T over k_0. Since $k_0(k^{p^q})/k_0$ is separably generated $\bar{k}_0(k^{p^q})/\bar{k}_0$ is separable. Separable extensions are p-independence preserving so T is a p-independent subset of k_1/k_0. Since $\bar{k}_0 \subset k_0(k^{(q+t)}) = k_0(k_1^{(t)})$ for all t [3, Lemma 4, p. 289] it follows that $T \subset k_0(k_1^{(t)})$ for all t and we have $k_0(k_1^{(t)}) \not\subset k_0(k_1^p)$ for all $t > 0$. By Proposition 2, k_1 contains a maximal separable subfield s over k_0 which is in $k_0(k_1^{(t)})$. By Proposition 3, if s_1 is a maximal separable subfield of k/k_0 containing s then $[k : s_1] \geqslant [k : k_0]_i p^t$. This establishes the result in one direction since t can be chosen arbitrarily large.

To show the converse we note that if $\bar{k}_0/k_0$ is separable then, for large q, $k_0(k^{(q)})/k_0 = \bar{k}_0(k^{p^q})/k_0$ is separable. Thus, if s is a maximal separable subfield of k/k_0 and $[k : s] = [k : k_0]_i p^q$ for q as above then, by Proposition 1, $s \subset k_0(k^{(q)})$ and so $s = k_0(k^{(q)})$. This leads to a contradiction since $\bar{k}_0(k^{p^q})$ is contained in every distinguished subfield. It follows that $[k : s] < [k : k_0]_i p^n$ where n is the least integer j such that $k_0(k^{(j)}) = \bar{k}_0(k^{p^j})$ and $k_0(k^{p^j})/k_0$ is separable.

The following remarks do not require the assumption tr.deg.$(k/k_0) = 1$. If s is a distinguished subfield of k/k_0 then $k_0(s^{p^n})$ is a distinguished subfield of $k_0(k^{(n)})$ and, so, if $k_0(k^{(n)})/k_0$ is separable, $k_0(s^{p^n}) \subset k_0(k^{p^n}) \subset k_0(k^{(n)}) = k_0(s^{p^n})$. Thus, $k_0(k^{(n)}) = \bar{k}_0(k^{p^n}) = k_0(k^{p^n})$ since $\bar{k}_0 \subset k_0(k^{(n)})$ for all n [3, Theorem 5, p. 289]. Thus $k_0(k^{(n)})/k_0$ is separable for some n if and only if $\bar{k}_0/k_0$ is separable, and if $k_0(k^{(n)})/k_0$ is separable then $k_0(k^{(n)}) = k_0(k^{p^n})$.

COROLLARY 1. *If* tr.deg.$(k/k_0) = 1$ *the following are equivalent.*

(i) *$k_0(k^{(n)})/k_0$ is separable for some n.*

(ii) *$\bar{k}_0/k_0$ is separable.*

(iii) $\{[k : s] \| s$ *a maximal separable subfield of* $k/k_0\}$ *is bounded.*

If one of the three conditions holds and s *is a maximal separable subfield of* k/k_0 *then* $[k : s] < [k : k_0]_i p^n$ *where* n *is the least integer such that* $k_0(k^{(n)})/k_0$ *is separable.*

The following result holds for arbitrary transcendence degree > 0.

COROLLARY 2. *If* $\bar{k}_0/k_0$ *is inseparable then* $\{[k : s] \| s$ *a maximal separable subfield of* $k/k_0\}$ *is unbounded.*

PROOF.[1] Let $\{x_1, \ldots, x_l\}$ be a transcendency basis for a distinguished subfield of k/k_0. Replace k_0 with $k_1 = k_0\{x_1, \ldots, x_{l-1}\}$. We assume $l > 1$ having the result for $l = 1$. If $\bar{k}_0/k_0$ is inseparable then $\bar{k}_1/k_1$ is inseparable and $\{[k : s] \| s$ is a maximal separable subfield of $k/k_1\}$ is unbounded by Theorem 1. It remains to show that if s is a maximal separable subfield of k/k_1 it is a maximal separable subfield of k/k_0. By [4, Proposition 2.4] we have $k^p \cap s \subset k_1(s^p)$ or $k^p \cap s \subset k_1(s^p) \cap k^p = k_0(x_1, \ldots, x_{l-1})(s^p) \cap k^p$. Since $\{x_1, \ldots, x_{l-1}\}$ is a p-independent subset of k/k_0 it follows that $k^p \cap s \subset k_0(x_1^p, \ldots, x_{l-1}^p)(s^p) \subset k_0(s^p)$. Hence, by statement (C), s is a maximal separable subfield of k/k_0.

REFERENCES

1. J. Deveney and J. Mordeson, *Distinguished subfields*, preprint.

2. J. Dieudonné, *Sur les extensions transcendantes*, Summa Brasil. Math. **2** (1947), 1–20. MR **10**, 5.

3. N. Heerema, *pth powers of distinguished subfields*, Proc. Amer. Math. Soc. **55** (1976), 287–291.

4. N. Heerema and T. Morrison, *Maximal separable subfields of finitely generated extensions*, preprint.

5. H. Kraft, *Inseparable Körpererweiterungen*, Comment. Math. Helv. **45** (1970), 110–118. MR **41** #5333.

DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306

---

[1] The referee has observed that Corollary 2 can be proved in a way which avoids the use of Propositions 2 and 3 by first proving the following. PROPOSITION. *If* k/k_0 *is finitely generated and has maximal separable subfields of unbounded codimension the same is true for any finite extension of* k.

Corollary 2 is then reduced to the case $k = k_0(x_1, \ldots, x_n)(b)$ where the x_i are indeterminates and b^p is in k_0, not in k_0^p.