THE BRUHAT ORDER OF THE SYMMETRIC GROUP 
IS LEXICOGRAPHICALLY SHELLABLE

PAUL H. EDELMAN

Abstract. The title theorem is proven. It then follows from a theorem of Björner that the simplicial complex of chains of this Bruhat order is shellable and thus Cohen-Macaulay. It is further established that this complex is a double cone over a triangulation of a sphere.

In this note we present an elementary proof that the Bruhat order of the symmetric group $S_n$ is lexicographically shellable and hence Cohen-Macaulay. Using a theorem of Verma we obtain as a corollary that $\Delta(S_n)$, the simplicial complex of chains of $S_n$, is a double cone over a triangulation of a sphere of dimension $(n-2)$. We will employ the notation and terminology of Björner [2].

A finite poset $P$ is said to be bounded if it has a maximum and a minimum element, denoted $\hat{1}$ and $\hat{0}$ respectively. It is called pure if all of its maximal chains are the same length and it is graded if it is both bounded and pure. The rank of $P$ is the length of a maximal chain. An element $x$ of a graded poset $P$ has a well-defined rank $\rho(x)$ equal to the length of an unrefinable chain from $\hat{0}$ to $x$ in $P$. If $P$ is bounded let $\bar{P}$ be the poset $P - \{\hat{0}, \hat{1}\}$.

The order complex $\Delta(P)$ of a poset $P$ is the simplicial complex of all chains in $P$. A poset is said to be shellable if $\Delta(P)$ is shellable. For the definition of a shellable complex see [2] or [4]. Similarly $P$ is called Cohen-Macaulay if $\Delta(P)$ is. See [1], [2] or [6] for the definition and significance of a Cohen-Macaulay complex.

Let $C(P)$ be the set of covering relations

$$C(P) = \{ (x, y) \in P \times P | x \text{ is covered by } y \}.$$

An edge-labeling of $P$ is a map $\lambda: C(P) \to \Lambda$ where $\Lambda$ is some poset. An edge-labeling corresponds to an assignment of elements of $\Lambda$ to the edges in the Hasse diagram of $P$. An unrefinable chain $x_0 < x_1 < \cdots < x_n$ in a poset with an edge-labeling $\lambda$ will be called increasing if $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{n-1}, x_n)$ in $\Lambda$.

With every saturated chain $c$, say with elements $x_0 < x_1 < \cdots < x_n$ of a poset $P$ having an edge-labeling $\lambda$, we associate the $n$-tuple

$$\pi(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{n-1}, x_n)).$$

We call $\pi(c)$ the Jordan-Hölder (J-H) sequence of $c$. Totally order $\Lambda^n$ by the lexicographic order: $(a_1, a_2, \ldots, a_n)$ precedes $(b_1, \ldots, b_n)$ if and only if $a_i < b_i$ in the first coordinate where they differ.
Let $\lambda$ be the edge-labeling of a graded poset $P$. $\lambda$ is said to be an $L$-labeling if it satisfies the following two conditions:

(i) In every interval $[x, y]$ of $P$ there is a unique increasing unrefinable chain $c$, $x = x_0 < x_1 < \cdots < x_n = y$.

(ii) The J-H sequence of the unique chain from (i) is lexicographically first among the J-H sequences of all unrefinable chains $x = z_0 < z_1 < \cdots < z_n = y$ in $[x, y]$.

A graded poset is called lexicographically shellable if there exists an $L$-labeling of $P$.

**Theorem (Björner [2]).** If $P$ is lexicographically shellable then $P$ is shellable and hence Cohen-Macaulay. □

We now define the Bruhat order on the symmetric group $S_n$. For our purposes $S_n$ will be the set of all permutations of the set $[n] = \{1, 2, \ldots, n\}$. We will write $\pi \in S_n$ as a word $a_1 a_2 \ldots a_n$ in the letters $1, 2, \ldots, n$. A reduction of $\pi$ is a permutation obtained from $\pi$ by interchanging some $a_i$ with some $a_j$ provided $i < j$ and $a_i > a_j$. Define $\sigma < \pi$ if $\sigma$ can be obtained from $\pi$ by a sequence of reductions. Figure 1 is a drawing of the poset $S_3$.

![Figure 1](image_url)

It is well known that the rank of a permutation $\pi$ in $S_n$ is the number of inversions in $\pi$, i.e. the number of pairs $(i, j)$ where $i < j$ and $a_i > a_j$. Thus if $\sigma$ is covered by $\pi$ then $\pi$ has one more inversion than $\sigma$. The rank of $S_n$ is $\binom{n}{2}$.

**Theorem.** $S_n$ is lexicographically shellable.

**Proof.** Let $Z$ be the set of ordered pairs $(i, j) \in [n] \times [n]$ such that $i < j$. Totally order $Z$ by $(i, j) < (r, s)$ if $i < r$ or if $i = r$ and $j' < s$. Let $\lambda: C(S_n) \rightarrow Z$ be the labeling

$$\lambda(\sigma_1, \sigma_2) = (i, j)$$

if $i$ and $j$ are interchanged in $\sigma_1$ to obtain $\sigma_2$ and $i < j$. For example in $S_3$ we have $\lambda(123, 213) = (1, 2)$ and $\lambda(213, 312) = (2, 3)$.

We proceed to show that $\lambda$ is an $L$-labeling by first showing that in any interval $[x, y]$ the lexicographically first chain increases and then showing that there is a unique increasing chain in $[x, y]$.
We will show that the lexicographically first chain increases by contradiction. There are a number of cases to consider. We will present one and leave the others to the reader. Let $c$ be the lexicographically first chain in $[x, y]$, $x = \pi_0 < \pi_1 < \cdots < \pi_n = y$ and suppose it has a decrease. Then there are three permutations $\pi_{r-1} < \pi_r < \pi_{r+1}$ such that

$$\lambda(\pi_{r-1}, \pi_r) > \lambda(\pi_r, \pi_{r+1}).$$

Suppose $\lambda(\pi_{r-1}, \pi_r) = (i, j)$ and $\lambda(\pi_r, \pi_{r+1}) = (i, k)$ where $j > k$. Then the permutation $\pi_{r-1}$ looks like

$$a_1a_2\ldots i\ldots j\ldots k\ldots a_n$$

since the interchange $(i, j)$ must produce a cover. Define $\pi'_r$ by interchanging $i$ and $k$ in $\pi_{r-1}$. $\pi'_r$ covers $\pi_{r-1}$ and is covered by $\pi_{r+1}$. Moreover $\lambda(\pi_{r-1}, \pi'_r) = (i, k)$. Since $(i, k) < (i, j)$ the chain $c'$ with $\pi'_r$ replacing $\pi_r$ in $c$ is lexicographically earlier than $c$. This is a contradiction.

There are other cases to consider when the labels $\lambda(\pi_{r-1}, \pi_r)$ and $\lambda(\pi_r, \pi_{r+1})$ are disjoint. They are similar to the above argument.

What is left to show is that the increasing chain in the interval $[x, y]$ is unique. This follows from a series of remarks. Let $x = a_1a_2\ldots a_n$ and $y = b_1b_2\ldots b_n$. Define $p(r) = j$ if and only if $r \sim a$ and define $s(r) = j$ if and only if $r = b$. Let $i$ be the smallest number such that $p(i) \neq s(i)$.

**Remark 1.** No number less than $i$ will appear in any label in $[x, y]$. Suppose this were not true. Then there is some label $(j, a_j)$ in $[x, y]$ where $j < i$ and $j$ is the smallest number appearing in such a label. Since $p(j) = s(j)$ when $y$ is interchanged with $a_j$, $j$ is moved to the right of the correct position for it in $y$. Since no smaller number appears as a label $j$ cannot be moved back to the left. Hence $j$ cannot appear in a label in $[x, y]$.

**Remark 2.** $p(i) < s(i)$. This follows from an argument similar to that used in Remark 1.

**Remark 3.** In an increasing chain, the first label contains $i$. The element $i$ must be switched sometime to get from $x$ to $y$ and since it is the smallest number it must occur first.

**Remark 4.** If $\lambda(x, \pi_1) = (i, k)$ where $\pi_1$ is the second permutation in an increasing chain, then $p(k) < s(i)$. This follows from the same arguments as those used in Remark 1.

Let $j$ be the smallest number such that $i < j$ and $p(i) < p(j) < s(i)$.

**Remark 5.** The first label on an increasing chain is $(i, j)$. Suppose this were not the case. By Remark 3 the first label involves $i$. Suppose it is $(i, k)$, $k \neq j$. By Remark 4, $p(k) < s(i)$. If $p(j) < p(k)$ then the switch $(i, k)$ increases the number of inversions in the permutation by at least two, since $k > j > i$. So $(i, k)$ does not produce a cover. Hence $p(k) < p(j)$. But sometime $i$ and $j$ must switch, since $p(i) < p(k) < p(j)$. Then the label $(i, j)$ appears which forces a decrease. So the first label must be $(i, j)$.

Hence an increasing chain is uniquely determined. Since the lexicographically first chain increases, $\lambda$ is an $L$-labeling and the proof is complete. \(\square\)
A graded poset is called Eulerian if in every interval \([x, y]\) the identity

\[
\sum_{x \leq z \leq y} (-1)^{\rho(z) - \rho(x)} = 0 
\]

holds. If \(P\) is Eulerian then \(\overline{P}\) also satisfies (\(\ast\)) for all intervals \([x, y]\). It is easily seen that every interval of rank 2 in an Eulerian poset is isomorphic to the poset in Figure 2. R. Stanley observed that if \(P\) satisfies (\(\ast\)) and is of rank \(k\) then \(\Delta(P)\) is a pseudomanifold of dimension \(k\), i.e. every \(k - 1\) face is contained in exactly two \(k\) faces. From this observation we deduce

**Corollary.** \(\Delta(S_n)\) is a triangulation of a sphere of dimension \((\frac{n}{2}) - 2\).

**Proof.** Verma [7] has shown that \(S_n\) is Eulerian. Hence \(\Delta(S_n)\) is a pseudomanifold. Since \(\Delta(S_n)\) is shellable by the previous theorem so is \(\Delta(S_n)\). Since it is known that a shellable pseudomanifold is a sphere (see for example [3, p. 444]) the corollary is proven.

R. Proctor has extended the Theorem to the Bruhat orders of the other classical Weyl groups as well as their quotients by parabolic subgroups [5].

![Figure 2](image)

**Figure 2**

**Note added in proof.** Björner and Wachs have recently generalized the Theorem for all Coxeter groups modulo a parabolic subgroup (Bruhat orders of Coxeter groups and shellability, Report 1980-No. 20, Department of Mathematics, University of Stockholm).

**References**

5. R. Proctor, Classical Bruhat orders are lexicographically shellable (in preparation).

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Current address: Department of Mathematics El, University of Pennsylvania, Philadelphia, Pennsylvania 19104

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use