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EQUIVALENCE OF CERTAIN REPRESENTING MEASURES

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ABSTRACT. If an interior component Ω of a compact $K \subset \mathbf{C}$ is a part for $R(K)$, then given z_1, z_2 in Ω and a representing measure λ_1 for z_1 there is a representing measure for z_2 equivalent to λ_1.

Given two points φ and ψ in the same Gleason part of a uniform algebra, and λ in M_φ (the set of representing measures for φ), a well-known result of Bishop [**2**, p. 143] insures there is a μ in M_ψ which bounds λ, and indeed there is a pair $\lambda \in M_\varphi$, $\mu \in M_\psi$ which mutually bound one another. However it is not in general the case that each λ in M_φ is equivalent to some element of M_ψ, and our purpose is to point out one instance where that occurs.

Recall that for $K \subset \mathbf{C}$ compact, $R(K)$ is the uniform closure in $C(K)$ of the rational functions.

THEOREM. *Suppose $K \subset \mathbf{C}$ is compact and Ω is a component of the interior K° which is also a Gleason part for $R(K)$. Then for $z_1, z_2 \in \Omega$ and $\lambda_1 \in M_{z_1}$ there is a $\lambda_2 \in M_{z_2}$ equivalent to λ_1; moreover for any nonzero measure μ on ∂K orthogonal to $R(K)$ with $\mu \ll \lambda_1$ there is a $\lambda \in M_{z_2}$ equivalent to $|\mu|$.*

Here M_z is the set of representing measures on ∂K, as usual. Of course the first conclusion fails if the part containing the component Ω contains a boundary point since no measure representing $z \in \Omega$ is equivalent to a point mass; and it could only extend to a part consisting of several interior components if these all shared the same boundary (since $\partial\Omega$ is always the topological support of harmonic measure for $z \in \Omega$). Consequently it seems unlikely that it holds for any noncomponent parts.

It is worth noting that the result is not due to all elements of M_z being mutually equivalent; an example is provided by the champagne bubble set K (given in [**5**, 27.6]) built from Beurling's function by McKissick in constructing his regular uniform algebra: there K consists of the unit disc D less disjoint subdiscs of finite total circumference converging to ∂D, K° is dense and connected, and there is an $f \in R(K)$ with $f^{-1}(0) = \partial D$. Consequently for $z \in K^\circ$, ∂D is necessarily a Jensen null set (so of harmonic measure λ_z zero); but via Cauchy dz provides an orthogonal measure μ on ∂K, which is absolutely continuous with respect to some $\lambda \in M_z$ by the known decomposition of orthogonal measures [**5**, 23.6] and Wilken's theorem [**2**, p. 47], and λ and λ_z are inequivalent.

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Our result is a simple consequence of an abstract F. and M. Riesz theorem for bands due to Brian J. Cole (unpublished) and König and Seever **[4]** which we state in a form convenient for our use (cf. [**1**, 3.1, p. 31]) for a uniform algebra A on X ($= \partial K$ in our application).

THEOREM (COLE-KÖNIG-SEEVER). *Let m be a probability measure on X and suppose $M_\varphi(m) = M_\varphi \cap L^1(m) \neq \varnothing$. If $\mu \ll m$ is a measure orthogonal to A and we choose*[2] *λ in $M_\varphi(m)$ so that $\|\mu_\lambda\|$ is a maximum, where μ_λ is the component of μ absolutely continuous with respect to λ, then $\mu_\lambda \perp A$.*

(Here $L^1(m)$ is our band [**1**], and $\mu = \mu_\lambda + \mu'_\lambda$, where μ'_λ is the component of μ singular with respect to λ, is our band decomposition of μ.)

We first apply this F. and M. Riesz theorem to the uniform algebra setting. (I am indebted to the referee for considerable simplification in what follows.)

PROPOSITION. *Let B be a uniform algebra on X, and let $g \in B$ be nonvanishing on X, with $1/g \in B$ while $C(X) = [B, 1/g]$, the closed algebra generated by B and $1/g$. Trivially then $A = \mathbf{C} + gB$ is a closed subalgebra of $C(X)$ with gB a maximal ideal, the kernel of a multiplicative linear functional φ on A.*

Suppose $\mu \perp B$, $\mu \neq 0$. Then there is a representing measure λ for φ which is equivalent to $|\mu|$.

To begin, since $\mu \neq 0$ and $C(X) = [B, 1/g]$, there is a least integer $n > 1$ with $\mu \not\perp g^{-n}B$, so $g^{-n}\mu \perp gB$ while $g^{-n}\mu(f) = 1$ for some $f \in B$. Thus $fg^{-n}\mu$ is a complex measure representing φ on $A = \mathbf{C} + gB$, and so dominates a true (i.e., $\geqslant 0$) representing measure λ for φ [**2**, p. 33], and $\lambda \in M_\varphi(|\mu|) = M_\varphi \cap L^1(|\mu|)$.

Now in $M_\varphi(|\mu|)$ we can find an element λ which is maximal in the sense that for any other element λ' we have $\lambda' \ll \lambda$, and this dominating λ clearly has the property that it maximizes $\|(b\mu)_\lambda\|$ for any fixed bounded function b, where $(b\mu)_\lambda$ is the component of $b\mu \ll \lambda$. So for $b \in B$ if we apply the Cole-König-Seever theorem to the measure $b\mu \perp A$ we have $(b\mu)'_\lambda = b\mu'_\lambda \perp A$, where the prime indicates the λ-singular component. In particular $\mu'_\lambda(b) = 0$ for all $b \in B$, so $\mu'_\lambda \perp B$.

But if $\mu'_\lambda \neq 0$, as in the first paragraph we obtain a representing measure λ' for φ on A with $\lambda' \ll |\mu'_\lambda|$, hence singular with respect to λ, and clearly $(\mu'_\lambda)_{\lambda'}$, the component of $\mu'_\lambda \ll \lambda'$, is nonzero; since

$$\|\mu_{(\lambda+\lambda')/2}\| \geqslant \|\mu_\lambda\| + \|(\mu'_\lambda)_{\lambda'}\| > \|\mu_\lambda\|$$

contradicts the maximality of $\|\mu_\lambda\|$ we conclude $\mu'_\lambda = 0$. Thus $\mu = \mu_\lambda \ll \lambda$, and λ and $|\mu|$ are equivalent as asserted.

COROLLARY. *Let $K \subset \mathbf{C}$ be compact and suppose the interior K° is connected and dense in K. Let μ be a measure on ∂K with $\mu \perp R(K)$, and let $z \in K^\circ$. Then there is a representing measure λ for z on $R(K)$ which is equivalent to $|\mu|$.*

Here we apply the proposition to $X = \partial K$ and the uniform algebra $B = R(K)$ on X, with $g(\zeta) = \zeta - z$. We have $A = R(K) = B$ and φ evaluation at z, and by

[2] If $\|\mu_{\lambda_n}\| \to$ maximum then $\lambda = \sum_1^\infty 2^{-n}\lambda_n$ provides such a maximizing λ.

Runge's theorem $[R(K), 1/g] = R(\partial K)$. Since every point of ∂K lies in the closure of the connected interior K° we know it is a peak point for $R(\partial K)$, and so $R(\partial K) = C(\partial K)$ by Bishop's theorem [2, p. 54]; thus the proposition applies to yield its corollary.

We can now obtain our theorem from the corollary. Note that its first assertion follows from the second by setting $\mu = (g - z_1)\lambda_{z_1}$; so we only need to see any μ on ∂K orthogonal to $R(K)$ with μ absolutely continuous with respect to a representing measure λ_0 for some point in Ω is equivalent, for any z in Ω, to some $\lambda \in M_z$. But our μ necessarily has, for $z \notin \Omega^-$, a Cauchy transform [2, p. 46] $\hat{\mu}(z)$ which must vanish: if $\hat{\mu}(z) \neq 0$ for $z \notin \Omega^-$ then $|\mu|$ dominates some representing measure λ for z, by an old result of Bishop, so since λ_0 and λ are not mutually singular, z would lie in the part Ω. Since $\hat{\mu}(z) = 0$ for all $z \notin \Omega^-$ implies $\mu \perp R(\Omega^-)$, while Ω^- necessarily has its interior a union of components of K°, hence just Ω, for $z \in \Omega$ we can apply the corollary to Ω^- to obtain a representing measure λ for z on $R(\Omega^-)$ equivalent to $|\mu|$. Trivially λ represents z on $R(K)$, so we are done.

In fact the theorem holds with $R(K)$ replaced by $A(K)$, or any T-invariant closed subalgebra [3] A_0 of $C(K)$ lying between $R(K)$ and $A(K)$; the proof is essentially the same since, for such an algebra, $A_0 = \mathbf{C} + (z - z_0)A_0$ (for $z_0 \in \Omega$), while K is the spectrum and ∂K the Šilov boundary. (In place of applying the corollary one can appeal to the proposition applied to $B = (A_1|\Omega^-)^- = \mathbf{C} + gB$, $g(\zeta) = \zeta - z$, noting that $[B, 1/g] \supset R(\partial\Omega) = C(\partial\Omega)$.)

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