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FIXED POINTS AND BOUNDARIES

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ABSTRACT. A lemma of Ludvik Janos is used to show that if a nonexpansive self-map T of a compact set X is contractive on $\Delta' X$, the boundary of X in $\overline{\text{co}}\, X$, then T has a fixed point in X. It is further proven that if $T(\Delta' X) \cap \Delta' X = \varnothing$, or if T maps any point y of X away from $\Delta' X$, then T has a fixed point in X.

In the results which follow, X is a compact subset of a strictly convex normed linear space E and $T: X \to X$ is nonexpansive. For a subset S of E we shall let $\Delta' S$ denote the boundary of S in $\overline{\text{co}}\, S$, the closed convex hull of S.

In [**1**, Lemma 1] we stated the following version of a result of Janos [**4**, Lemma 3.1]:

LEMMA 1. *There exists a nonexpansive retraction r of X onto the compact set $C_T = \bigcap_{i=1}^{\infty} T^i X$. Furthermore, this r is in the closure, in the pointwise topology of X^X, of the set $\{T^i\}_{i=1}^{\infty}$.*

LEMMA 2. $\Delta' C_T \subseteq \Delta' X$.

PROOF. We claim that $(\text{co}\, C_T) \cap X = C_T$, for if not, there exists an $x \in X \setminus C_T$ such that $x = a_1 x_1 + \cdots + a_n x_n$ where $x_i \in C_T$, $a_i > 0$, and $\sum_{i=1}^{n} a_i = 1$. Edelstein has shown [**2**, Proposition 2] that if $f: E \to E$ is a nonexpansive mapping of a strictly convex normed linear space E into itself and if $f|_A$, $A \subset E$, is an isometry, then

$$x = ax_1 + (1 - a)x_2 \quad \text{for } x_1, x_2 \in A \text{ and } 0 < a < 1$$

implies that

$$f(x) = af(x_1) + (1 - a)f(x_2).$$

Thus, since $r|_{C_T}$ is the identity map we have $rx = r(a_1 x_1 + \cdots + a_n x_n) = a_1 r x_1 + \cdots + a_n r x_n = a_1 x_1 + \cdots + a_n x_n = x$. But $rx \in C_T$ and so $x \in C_T$ which is impossible. Thus $(\text{co}\, C_T) \cap X = C_T$.

Now if $y \in \Delta' C_T$ and U is any open set containing y, then the open set $U \setminus C_T$ contains a point in $(\overline{\text{co}}\, C_T) \setminus C_T$, and so also a point z in $(\text{co}\, C_T) \setminus C_T$. But z cannot belong to X since $(\text{co}\, C_T) \cap X = C_T$. Thus $z \in (\overline{\text{co}}\, X) \setminus X$ and so $y \in \Delta' X$. Hence $\Delta' C_T \subseteq \Delta' X$.

THEOREM 1. *If $T|_{\Delta' X}$ is contractive (i.e. if $\|Tx - Ty\| < \|x - y\|$ for all x, y in $\Delta' X$) then T has a fixed point in X.*

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PROOF. T maps C_T onto C_T and C_T is compact. If C_T is convex, then Schauder's theorem implies that T has a fixed point in C_T. If C_T is not convex, then $\Delta' C_T \neq \varnothing$ and obviously $|\Delta' C_T| \geqslant 2$. By Lemma 2, $\Delta' C_T \subseteq \Delta' X$. Any nonexpansive mapping of a compact metric space onto itself is an isometry [3, Satz Ib] and so T is an isometry on C_T and thus on $\Delta' C_T$. This is a contradiction, and so C_T must be convex and thus contain a fixed point of T.

THEOREM 2. *If $T(\Delta' X) \cap \Delta' X = \varnothing$ then T has a fixed point in X.*

PROOF. As before, if C_T is convex, we are done. So suppose $\Delta' C_T \neq \varnothing$.

Now the retraction r is the identity map I on C_T, and so I is in the closure, in the pointwise topology, of the set $\{T^i\}_{i=1}^{\infty}$. This means that for any finite subset of C_T, in particular for any $x, y \in \Delta' C_T$, we can find a subsequence $\{n_j\}$ so that $T^{n_j}x \to Ix = x$ and $T^{n_j}y \to Iy = y$.

Let us assume then that $x, y \in \Delta' C_T$ and, without loss of generality, that the open line segment $L = \{ax + (1-a)y | 0 < a < 1\}$ lies entirely in the complement of C_T. Since Tx and Ty do not belong to $\Delta' C_T$ (for by Lemma 2, $\Delta' C_T \subseteq \Delta' X$) there exists an a, $0 < a < 1$, such that $z = aTx + (1-a)Ty$ belongs to C_T. Let $\{n_j\}$ be the subsequence associated with x, y such that $T^{n_j}x \to x$ and $T^{n_j}y \to y$. Then by [2, Proposition 2], $T^{n_j-1}z = T^{n_j-1}(aTx + (1-a)Ty) = aT^{n_j}x + (1-a)T^{n_j}y$ which approaches the point $ax + (1-a)y$ on L in the complement of C_T. This is impossible since $T\colon C_T \to C_T$. Thus C_T is convex and so contains a fixed point of T.

(Note that in the proof of this theorem we could have assumed as the hypothesis that if there exists an $n \geqslant 1$ such that $T^n \Delta' X \cap \Delta' X = \varnothing$ then T has a fixed point in X.)

Given a subset $Q \subset X$ and a point $y \in X$ we say that "T maps y away from Q" if $\|Ty - x\| > \|y - x\|$ for all $x \in Q$.

THEOREM 3. *If there exists a point $y \in X$ such that T maps y away from $\Delta' X$, then T has a fixed point in X.*

PROOF. The real-valued map $\phi(x) = \|Ty - x\| - \|x - y\| > 0$ on the compact set $\Delta' X$ is continuous and so there exists an $\varepsilon > 0$ such that $\|Ty - x\| - \|y - x\| > \varepsilon$ for all $x \in \Delta' X$. If $\Delta' C_T \neq \varnothing$, there is a point z in the compact set $\Delta' C_T$ such that $r = \|y - z\| = \operatorname{dist}(y, \Delta' C_T)$. Consider the open sphere $S = S(y, r + \varepsilon)$, and let $w \in S \cap \{\operatorname{co} C_T \setminus C_T\}$. On the closed line segment $[wz]$ there exists a point $z_0 \in \Delta' C_T$ such that the open line segment (wz_0) lies entirely in $\operatorname{co} C_T \setminus C_T$.

Obviously $Tz_0 \in C_T \cap S(Ty, r + \varepsilon)$. We claim that $S(Ty, r + \varepsilon) \cap \Delta' C_T = \varnothing$, for if $x \in \Delta' C_T$ then $\|y - x\| \geqslant r$ and $\|Ty - x\| - \|y - x\| > \varepsilon$ so that $\|Ty - x\| > r + \varepsilon$. Thus $Tz_0 \in C_T$ but $Tz_0 \notin \Delta' C_T$.

Let us suppose that the w above is represented: $w = a_1x_1 + \cdots + a_kx_k$ where $x_i \in C_T$, $\sum_{i=1}^k a_i = 1$ and $a_i > 0$. The points Tx_i all lie in C_T and thus the point $w_0 = a_1Tx_1 + \cdots + a_kTx_k$ is in $\operatorname{co} C_T$. Now on the open line segment (w_0Tz_0) there must exist a point $p \in C_T$, for otherwise $Tz_0 \in \Delta' C_T$ which is impossible. Let $p = aTz_0 + (1-a)w_0$ where $0 < a < 1$. Then $p = aTz_0 + \sum_{i=1}^k (1-a)a_iTx_i$ and $a + \sum_{i=1}^k (1-a)a_i = 1$.

For the points $z_0, x_1, \ldots, x_k$ let $\{n_j\}$ be a subsequence of $\{n\}$ for which $T^{n_j}z_0 \to z_0$ and $T^{n_j}x_i \to x_i$, $1 \leq i \leq k$. (We can do this since the identity on C_T is in the pointwise closure of $\{T^n\}_{n=1}^{\infty}$.) Then by [2, Proposition 2],

$$T^{n_j-1}p = T^{n_j-1}\left[aTz_0 + \sum_{i=1}^{k}(1-a)a_iTx_i\right]$$

$$= aT^{n_j}z_0 + \sum_{i=1}^{k}(1-a)a_iT^{n_j}x_i$$

which approaches the point $az_0 + (1-a)\sum_{n=1}^{k} a_ix_i = az_0 + (1-a)w$. Thus $T^{n_j-1}p$ approaches a point lying on the open line segment (wz_0). This is not possible since (wz_0) lies entirely outside C_T and $T\colon C_T \to C_T$. Thus $\Delta' C_T = \varnothing$, C_T is convex, and so T has a fixed point in C_T.

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