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A FIXED POINT FREE NONEXPANSIVE MAP

DALE E. ALSPACH[1]

ABSTRACT. In this note we give an example of a weakly compact convex subset of $L_1[0, 1]$ that fails to have the fixed point property for nonexpansive maps. This answers a long-standing question which was recently raised again by S. Reich [7].

1. Introduction. A (usually nonlinear) map T on a subset K of a Banach space X is said to be nonexpansive if for every k_1, k_2 in K, $\|Tk_1 - Tk_2\| \leqslant \|k_1 - k_2\|$. Many authors have given conditions on the set K that guarantee that a nonexpansive map T on K has a fixed point, e.g., [1], [2], [5], [6]. Usually K is assumed to be weakly compact and convex. Of course, if T is weakly continuous, then T has a fixed point by the Schauder-Tychonoff fixed point theorem. For T nonexpansive, (and not weakly continuous) positive results have been obtained only by placing additional requirements on K; however, it was unknown whether any of these additional requirements on K were necessary. Our example shows that in fact some additional assumptions on K are necessary.

2. The example. Let $X = L_1[0, 1]$ and let

$$K = \left\{ f \in L_1[0, 1]: \int_0^1 f = 1, 0 \leqslant f \leqslant 2, \text{ a.e.} \right\}.$$

It is easy to see that K is a weakly closed, convex subset of the order interval $\{f\colon 0 \leqslant f \leqslant 2\}$, and thus K is weakly compact, because order intervals in $L_1[0, 1]$ are weakly compact. (This is a direct consequence of uniform integrability, [3, p. 292].) Define the map T from K to K by

$$Tf(t) = \begin{cases} 2f(2t) \wedge 2, & 0 \leqslant t \leqslant \frac{1}{2}, \\ [2f(2t-1) - 2] \vee 0, & \frac{1}{2} < t \leqslant 1. \end{cases}$$

(We will use equality throughout with the understanding that there may be an exceptional set of measure zero.) We leave it to the reader to check that T is an *isometry* on K.

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Suppose that T has a fixed point g. We note first that $g = 21_A$ for some set A of measure one-half. Indeed,

$$\begin{aligned}\{t\colon g(t) = 2\} &= \{t\colon Tg(t) = 2\} \\ &= \{t/2\colon g(t) = 2\} + \left\{\frac{1+t}{2}\colon g(t) = 2\right\} \\ &\quad + \{t/2\colon 1 \leq g(t) < 2\}.\end{aligned}$$

(We are using + to denote disjoint union.) Because the measure of $\{t/2\colon g(t) = 2\} + \{(1+t)/2\colon g(t) = 2\}$ is equal to the measure of $\{t\colon g(t) = 2\}$, it follows that $\{t\colon 1 \leq g(t) < 2\}$ is of measure zero. Iteration of this argument shows that

$$\{t\colon 0 < g(t) < 2\} = \bigcup_{n=0}^{\infty} \{t\colon 2^{-n} \leq g(t) < 2^{-n+1}\}$$

is of measure zero, as well.

Next observe that for $g = 21_A$

$$\{t\colon T^n g(t) = 2\} = \sum_{\varepsilon_i \in \{0,1\}} \left\{\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2^2} + \cdots + \frac{\varepsilon_n}{2^n} + \frac{t}{2^n}\colon t \in A\right\}$$

for all n. We have this for $n = 1$ above, and induction establishes it in general. Because g is fixed, $A = \{t\colon T^n g(t) = 2\}$ for all natural numbers n and thus, the intersection of A with any interval with dyadic end points has measure exactly half the measure of the interval. Obviously no such measurable set exists. This contradiction shows that T has no fixed point.

REMARK 1. The set K has diameter two, but $\|f - 1\| \leq 1$ for all $f \in K$ and thus, K cannot be the minimal weakly compact convex subset invariant under T. In particular, the set

$$\bigcap_{i=1}^{\infty} \{f\colon \|f - (1 + r_i)\| \leq 1\} \cap \{f\colon \|f - 1\| \leq 1\} \cap K,$$

where $r_i = \operatorname{sgn}[\sin 2\pi i t]$, the ith Rademacher function, is invariant.

REMARK 2. It remains open whether there is a closed, bounded, convex subset of a reflexive space (hence, weakly compact) without the fixed point property for nonexpansive maps.

REMARK 3. When viewed as a transformation acting on the sets $\{(x, y)\colon 0 \leq y \leq f(x)\}$. This example is essentially the baker's transformation from ergodic theory [4]. The various properties of our example can be derived from the well-known properties of that transformation.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74074