THE SEPARATION PRINCIPLE
FOR IMPULSE CONTROL PROBLEMS

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ABSTRACT. In this paper, one shows that the combined problem of optimal impulse control and filtering, for a stochastic linear dynamic system observed via a noisy linear channel, can be reduced to two independent problems of impulse control and filtering, respectively.

1. Introduction. W. M. Wonham [8] showed that the combined problem of optimal control and filtering, for a stochastic linear dynamic system observed via a noisy linear channel, can be reduced to two independent problems of stochastic control and filtering, respectively. This result was improved by M. H. A. Davis [3] using the concept of Girsanov solutions of stochastic differential equations.

A. Bensoussan and J. L. Lions [1] proved that the same separation principle holds for stopping time problems.

In all cases, a nondegeneracy on the observation matrix is imposed. This assumption would rarely be met in practice.

In [5], we showed that the separation principle for stopping time problems holds even under degeneracy.

Let us also mention the work of J. Szpirglas and G. Mazziotto [7].

The object of this article is to prove that the combined problem of optimal impulse control and filtering, for a stochastic linear dynamic system observed via a noisy linear channel, can be reduced to two independent problems of impulse control and filtering, respectively. In general, the optimal impulse control depends parametrically on the intensity of channel noise; the result means, however, that channel noise plays qualitatively the same role as dynamic disturbances in determination of the feedback law.

2. Statement of the problem. Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(T\) be a positive constant.

Given matrices \(F(t), G(t), H(t), 0 < t < T\), such that

\[
\begin{align*}
F(\cdot), G(\cdot) & \in C([0, T]; \mathbb{R}^n \times \mathbb{R}^n), \\
H(\cdot) & \in C([0, T]; \mathbb{R}^n \times \mathbb{R}^p),
\end{align*}
\]

(2.1)

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we denote by $y^o(t)$ the solution of the linear Itô equation
\[
\begin{align*}
\frac{dy^o(t)}{dt} &= F(t)y^o(t) + G(t)dw(t), & 0 < t < T, \\
y^o(0) &= x + \xi, & x \in \mathbb{R}^N,
\end{align*}
\] (2.2)

where $w(t)$ is a standard Wiener process in $\mathbb{R}^N$ and $\xi$ is a Gaussian random variable with vanishing expectation and covariance matrix $P_0$; $\xi$ is independent of the process $w(t), 0 < t < T$.

The current state of the system without control at the instant $t$ is $y^o(t)$, but we cannot observe the system. The information is provided by the channel output $z^o(t)$ defined by
\[
\begin{align*}
\frac{dz^o(t)}{dt} &= H(t)y^o(t) + d\eta(t), & 0 < t < T, \\
z^o(0) &= 0,
\end{align*}
\] (2.3)

where $\eta(t)$ is a Wiener process in $\mathbb{R}^P$ independent of $w(t)$, with vanishing expectation and covariance matrix $R(t)$ such that
\[
\begin{align*}
R(\cdot) &\in C([0, T]; \mathbb{R}^P \times \mathbb{R}^P), \\
R(t) &> rI, & r > 0 \quad \forall t \in [0, T].
\end{align*}
\] (2.4)

We denote by $\mathcal{Z}^o$, $0 < t < T$, the nondecreasing right continuous family of completed $\sigma$-algebras generating by the process $z^o(t)$.

An admissible impulse control $v$ is a set $\{\theta_1, \xi_1, \ldots ; \theta_n, \xi_n, \ldots \}$ where $\{\theta_i\}_{i=1}^{\infty}$ is an increasing sequence of stopping times with respect to $\mathcal{Z}^o$ convergent to $T$ ($0 < \theta_1 < \theta_1 + \ldots < T, \{\theta_i, \xi_i\} \in \mathcal{Z}^o, \theta_i \to T$) and $\{\xi_i\}_{i=1}^{\infty}$ is a sequence of random variables taking values in $\mathbb{R}^N_+$, adapted with respect to $\{\theta_i\}_{i=1}^{\infty}$ ($\xi_i: \Omega \to \mathbb{R}^N, \xi_i > 0$, $\mathcal{Z}^o$-measurable).

Now we define the sequence of diffusion processes with jumps, $\{y^n(t)\}_{n=1}^{\infty}$, $y^n(t) = y^n(t, v), t \in [0, T], \forall$ any admissible impulse control, by the stochastic equation
\[
\begin{align*}
\frac{dy^n(t)}{dt} &= F(t)y^n(t) + G(t)dw(t), & \theta_n < t < T, \\
y^n(t) &= y^{n-1}(t) + 1_{\theta_n = \xi_n}, & 0 < t < \theta_n. 
\end{align*}
\] (2.5)

We have
\[
y^n(t) = y'^i(t) \quad \text{on } [0, \theta_n], \forall i > n. \tag{2.6}
\]

Defining
\[
y(t, v) = \lim_{n \to \infty} y^n(t), & 0 < t < T, \tag{2.7}
\]
the process $y(t, v)$, which is right continuous with left limits existing, satisfies the following stochastic equation:
\[
\begin{align*}
\frac{dy(t)}{dt} &= F(t)y(t)dt + G(t)dw(t) + \sum_{i=1}^{\infty} \xi_i \delta(t - \theta_i)dt, & 0 < t < T, \\
y(0) &= x + \xi,
\end{align*}
\] (2.8)

where $\delta(t)$ is the Dirac measure.

\footnote{$1_{\theta_n = \xi}$ denotes the characteristic function of the set $\{\theta_n = \xi\}$.}
The current state of the system with impulse control \( v \) at the instant \( t \) is represented by \( y(t) \), and
\[
\dot{y}(t) = E\{ y(t) / \mathcal{F}_t \}
\]
is the information state process; we also have \( \dot{y}(0) = x \).

We call the impulse process \( \beta(t) \) the solution of the equation
\[
\begin{align*}
d\beta(t) &= F(t)\beta(t)dt + \sum_{i=1}^{\infty} \xi_i \delta(t - \theta_i)dt, \quad 0 < t < T, \\
\beta(0) &= 0.
\end{align*}
\]
Clearly, \( \beta(t) = \beta(t, v) \) is built in the same way as \( y(t) \) by iteration. Notice, the process \( \beta(t) \) is right continuous with left limits and adapted to the observation \( \mathcal{F}_t \).

Thus, according to the equation (2.2), (2.8), (2.10) we deduce from (2.9)
\[
\dot{y}(t) = E\{ y^o(t) / \mathcal{F}_t \} + \beta(t).
\]

We introduce the process \( e(t) \), called the estimation error, given by
\[
e(t) = y^o(t) - E\{ y^o(t) / \mathcal{F}_t \}, \quad 0 < t < T,
\]
which is independent of \( \mathcal{F}_t \) and verifies
\[
e(t) = y(t) - \dot{y}(t), \quad 0 < t < T.
\]

We also define \( \hat{\eta}(t) \) by
\[
\begin{align*}
d\hat{\eta}(t) &= R^{-1/2}(t)H(t)\epsilon(t)dt + R^{-1/2}(t)d\eta(t), \quad 0 < t < T, \\
\hat{\eta}(0) &= 0
\end{align*}
\]
which is a standard Wiener process and satisfies the martingale property
\[
\hat{\eta}(t) = E\{ \hat{\eta}(s) / \mathcal{F}_t \}, \quad 0 < t < s < T.
\]

Then, the assertions (2.10), (2.11) and the R. E. Kalman-R. S. Bucy [4] theory show that \( \dot{y}(t) \) is the solution of the following stochastic equation
\[
\begin{align*}
d\dot{y}(t) &= F(t)\dot{y}(t)dt + \sum_{i=1}^{\infty} \xi_i \delta(t - \theta_i)dt, \quad 0 < t < T, \\
\dot{y}(0) &= x,
\end{align*}
\]
where the matrix \( P(t) \) is the unique solution of the Riccati equation
\[
\begin{align*}
P'(t) &= FP + PF^* - PH^*R^{-1}HP + GG^*, \quad 0 < t < T, \\
P(0) &= P_0.
\end{align*}
\]

We also deduce that the estimation error \( \epsilon(t) \) is the unique solution of the Itô equation
\[
\begin{align*}
de(t) &= (F - PH^*R^{-1}H)\epsilon(t)dt - PH^*R^{-1}d\eta + Gd\omega, \quad 0 < t < T, \\
\epsilon(0) &= \xi.
\end{align*}
\]

\footnote{The prime (') means time derivative and the star (*) denotes the transpose.}
3. Optimal impulse control. Let $f(x, t)$ be a nonnegative, continuous and bounded function on $\mathbb{R}^N \times [0, T]$ taking values in $\mathbb{R}$,
\[
f \in C_b\left(\mathbb{R}^N \times [0, T]\right), \quad f(x, t) > 0 \quad \forall x \in \mathbb{R}^N, \quad t \in [0, T],
\]
and let $k(\xi)$ be a continuous function from $\mathbb{R}^N$ into $\mathbb{R}$ such that
\[
k \in C\left(\mathbb{R}^N\right), \quad k(\xi) > k_0 > 0, \quad k(\xi) \to \infty \quad \text{if} \quad |\xi| \to \infty.
\]
Now, for any admissible impulse control $\nu = \{\theta_1, \xi_1; \ldots; \theta_n, \xi_n; \ldots\}$ and $x \in \mathbb{R}^N$ we set
\[
J_x(\nu) = \mathbb{E}\left\{ \int_0^T f(y(t), t) e^{-\alpha t} dt + \sum_{i=1}^{\infty} k(\xi_i) 1_{\Delta t < T} e^{-\alpha \Delta t} \right\},
\]
where $\alpha$ is a real constant.

We remark that any admissible impulse control $\nu$ is adapted to the information state $\dot{y}(t)$ and not to the current state $y(t)$.

Our purpose is to characterize the optimal cost
\[
u_0(x) = \inf\{J_x(\nu)/\nu \text{ admissible impulse control}\}
\]
and to obtain a separation principle for an eventual optimal admissible impulse control.

Let $M$ be the operator
\[
[M\phi](x) = \inf\{k(\xi) + \phi(x + \xi)/\xi \in \mathbb{R}^N\}
\]
and $u(x, t)$ be an arbitrary function satisfying
\[
u \in C_b\left(\mathbb{R}^N \times [0, T]\right), \quad u < Mu \quad \text{in} \quad \mathbb{R}^N \times [0, T].
\]
The admissible impulse control $\nu = \nu_x$ associated to the function $u$ is defined as follows. First we select a function $\xi(x, t)$ verifying
\[
\left\{
\begin{array}{c}
\xi: \mathbb{R}^N \times [0, T] \to \mathbb{R}^N, \text{Borel measurable and bounded such that} \\
[Mu](x, t) = k(\xi(x, t)) + u(x + \xi(x, t), t) \quad \forall x \in \mathbb{R}^N, t \in [0, T].
\end{array}
\right.
\]
Next, define $\tilde{\theta}^o = 0$ and $\tilde{\phi}^o(t)$ by
\[
\left\{
\begin{array}{c}
d\tilde{\phi}^o(t) = F(t)\tilde{\phi}^o(t)dt + P(t)H^*(t)R^{-1/2}(t)d\tilde{w}(t), \quad 0 < t < T, \\
d\tilde{\phi}^o(0) = x.
\end{array}
\right.
\]
We define $\nu = \{\theta_1, \xi_1; \ldots; \theta_n, \xi_n; \ldots\}$ by the formulas
\[
\tilde{\theta}^{i+1} = \inf\{t \in [\tilde{\theta}^i, T]/u(\tilde{\phi}^i(t), t) = [Mu](\tilde{\phi}^i(t), t)\}, \quad i = 0, 1, \ldots,
\]
\[
\left\{
\begin{array}{c}
\theta_i = \begin{cases}
\tilde{\theta}^i & \text{if} \quad \tilde{\theta}^i < T, i = 1, 2, \ldots, \\
T & \text{otherwise,}
\end{cases} \\
\xi_i = \xi(\tilde{\phi}^{i-1}(\theta_i), \theta_i), \quad i = 1, 2, \ldots.
\end{array}
\right.
\]
We define $\tilde{\phi}^i(t) = F(t)\tilde{\phi}^i(t)dt + P(t)H^*(t)R^{-1/2}(t)d\tilde{w}(t)$,
\[
\theta_i < t < T, \quad i = 1, 2, \ldots,
\]
\[
\tilde{\phi}^i(t) = \tilde{\phi}^{i-1}(t) + 1_{t = \theta_i}\xi_i, \quad 0 < t < \theta_i.
\]
Clearly, if there exists a function \( u \) verifying (3.6) whose associated admissible impulse control \( v \) is optimal, the separation principle is established. Notice, the fact that \( v \) is optimal shows automatically that \( \theta_i \to T \). Moreover, \( \theta_i = T \) for all \( i \geq n(\omega) \) almost surely.

Let \( A(t) \) be the second order differential operator corresponding to the Itô equation (3.8),

\[
A(t) = - \sum_{i,j=1}^{N} a_{ij}(t) \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{N} (F(t)x)_i \frac{\partial}{\partial x_i} + \alpha, \tag{3.13}
\]

where

\[
[a_{ij}(t)]_{ij} = \frac{1}{2} P(t) H^*(t) R^{-1}(t) H(t) P(t). \tag{3.14}
\]

We remark that \( A(t) \) is usually degenerate. W. M. Wonham [8], M. H. A. Davis [3], A. Bensoussan and J. L. Lions [1] supposed that the matrices \( P(t) \) and \( H(t) \) are nonsingular.

We set

\[
l(x, t) = E\{f(x + e(t), t)\} \quad \forall x \in \mathbb{R}^N, t \in [0, T], \tag{3.15}
\]

where \( e(t) \) is given by (2.18).

We introduce the following quasi-variational inequality. Find \( u(x, t) \) such that

\[
\begin{cases}
 u \in C_b(\mathbb{R}^N \times [0, T]), & u(x, T) = 0 \forall x \in \mathbb{R}^N, \\
 -\frac{\partial u}{\partial t} + A(t)u < l & \text{in } \mathcal{D}'(\mathbb{R}^N \times [0, T]), u < Mu & \text{in } \mathbb{R}^N \times [0, T], \\
 -\frac{\partial u}{\partial t} + A(t)u = l & \text{in } \mathcal{D}'([u \leq Mu]).
\end{cases} \tag{3.16}
\]

We have the

**Separation Principle Theorem.** Let the assumptions (2.1), (2.4), (3.1), (3.2) hold. Then there exists one and only one solution \( u \) of the quasi-variational inequality (3.16). Moreover the admissible impulse control \( v \) defined by (3.7)–(3.12), associated to the function \( u \) given by (3.16), is optimal [i.e., \( u_0(x) = J_x(v_x) \)].

**Proof.** First, using a general result in [6] applied to a degenerate operator \(-\partial/\partial t + A(t)\), we deduce that there exists a solution of problem (3.16).

In order to prove the uniqueness, we denote by \( z(s) = z_{x_0}(s, \omega), 0 < t < s < T, x \in \mathbb{R}^N, \omega \in \Omega, \) the diffusion associated to the operator \(-\partial/\partial t + A(t)\), i.e.,

\[
\begin{cases}
 dz(s) = F(s)z(s)ds + P(s)H^*(s)R^{-1/2}(s)d\mathcal{W}(s), & t < s < T, \\
 z(t) = x.
\end{cases} \tag{3.17}
\]

Now let \( u(x, t) \) be an arbitrary solution of (3.16). We set \( \theta = \theta_x(\omega), 0 < t < T, x \in \mathbb{R}^N, \omega \in \Omega, \) the first exit time of process \( z(s) \) from \([u \leq Mu]\), i.e.,

\[
\theta = \inf\{ s \in [t, T] / u(z(s), s) = [Mu](z(s), s) \}. \tag{3.18}
\]
Then, using the fact that the coefficients of the second order terms of operator $A(t)$ are constant and that $u(x, t)$ is continuous, we establish by convolution techniques the following Itô formulas for each $x \in \mathbb{R}^N$, $t \in [0, T]$:

$$u(x, t) \leq E \left\{ \int_t^{T \land \tau} l(z(s), s)e^{-as}ds + u(z(T \land \tau), T \land \tau)e^{-a(T \land \tau)} \right\}$$

$\forall \tau \geq t$ stopping time, \hspace{1cm} (3.19)

$$u(x, t) = E \left\{ \int_\theta^\tau l(z(s), s)e^{-as}ds + u(z(\theta), \theta)e^{-a\theta} \right\}.$$ \hspace{1cm} (3.20)

Therefore, as in [6], the properties (3.19), (3.20) imply the uniqueness of the solution $u$.

Next, from (3.7)-(3.12) and (3.20), we deduce

$$u(x, 0) = E \left\{ \int_0^T l(\hat{y}(t), t)e^{-at}dt + \sum_{i=1}^{\infty} k(\xi_i)1_{\xi_i \leq \tau}e^{-a\tau} \right\},$$ \hspace{1cm} (3.21)

and from (2.13), (3.15) we have

$$E \left\{ \int_0^T l(\hat{y}(t), t)e^{-at}dt \right\} = E \left\{ \int_0^T f(y(t), t)e^{-at}dt \right\};$$ \hspace{1cm} (3.22)

hence

$$u(x, 0) = J_x(\nu), \hspace{1cm} \nu \text{ associated to } u.$$ \hspace{1cm} (3.23)

Similarly, using (3.19), we obtain

$$u(x, 0) = \inf \{ J_x(\nu) / \nu \text{ admissible impulse control} \}.$$ \hspace{1cm} (3.24)

Then, (3.23) and (3.24) give

$$u(x, 0) = u_0(x), \hspace{1cm} \text{optimal cost (3.4)},$$ \hspace{1cm} (3.25)

and the theorem is proved. \hspace{1cm} □

**Remark 1.** If the function $f(x, t)$ is Lipschitz continuous, so is the function $u(x, t)$. In this case, $u$ is also the maximum solution of a classical quasi-variational inequality introduced by A. Bensoussan and J. L. Lions [2]. \hspace{1cm} □

**Remark 2.** This result can be extended for a function $k(\xi, x, t)$ instead of $k(\xi)$ appearing in the definition of cost (3.3). Clearly, we can replace the condition $\xi \in \mathbb{R}_+^N$ by $\xi \in \Lambda$, where $\Lambda$ is a closed subset of $\mathbb{R}_+^N$. \hspace{1cm} □

**Remark 3.** Using the technique presented in this paper, we can improve the result obtained in [5]. \hspace{1cm} □

**REFERENCES**


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