SPACES FOR WHICH ALL COMPACT METRIC SPACES ARE REMAINDERS

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Abstract. Let $X$ be a locally compact, completely regular, Hausdorff space, and let $K(X)$ be the lattice of compactifications of $X$. Conditions on $K(X)$ and an internal condition are obtained which characterize when $X$ has all compact metric spaces as remainders.

1. Introduction. Throughout this paper $X$ will denote a noncompact, locally compact, completely regular, Hausdorff space. A remainder of $X$ is any $\alpha X - X$, where $\alpha X$ is a Hausdorff compactification of $X$. One of the major concerns in the study of remainders has been the problem of characterizing when all members of a certain class of spaces can serve as remainders for each $X$ in another class of spaces (cf. [1], [2], [3], [7], [8], [11], [12], etc.). Let $K(X)$ denote the complete lattice of compactifications of $X$ (see [6]). The purpose of this paper is to determine those spaces for which all compact metric spaces are remainders. (Clearly, such spaces must be locally compact.) An internal characterization and a characterization in terms of $K(X)$ are obtained.

2. Characterization. In general, notation and terminology concerning remainders will follow that of [2]. For convenience, we shall say $X$ has a countable remainder whenever some $\alpha X - X$ is countably infinite. For $\alpha X$, $\gamma X \in K(X)$, we recall that $\alpha X \geq \gamma X$ if and only if there exists a continuous mapping of $\alpha X$ onto $\gamma X$ which is the identity on $X$. Let $\beta X$ denote the Stone-Cech compactification of $X$ and let $N$ denote the natural numbers.

Theorem 2.1. For locally compact $X$, the following are equivalent:

(A) There exists a chain $\{\alpha_n X | n \in N\}$ in $K(X)$, where $\alpha_n X - X = \{a_i^n | i = 1, \ldots, 2^n\}$ and where $\alpha_{n+1} X \geq \alpha_n X$ under mappings $t_{n+1}$ which satisfy $t_{n+1}(a_i^{n+1}) = t_{n+1}(a_i^n) = a_i^n$, for $i = 1, \ldots, 2^n$.

(B) Every compact metric space is a remainder of $X$.

(C) There exists a sequence of families $\mathcal{G}_n$ of pairwise disjoint, nonempty, open subsets of $X$ such that for each $n \in N$, $\mathcal{G}_n = \{G_i^n | i = 1, \ldots, 2^n\}$ and

(i) $G_i^{n+1} \cup G_{2i}^{n+1} \subseteq G_i^n$, $i = 1, \ldots, 2^n$,

(ii) $K_n = X - \bigcup \{G_i^n | i = 1, \ldots, 2^n\}$ is compact,

(iii) $K_n \cup G_i^n$ is noncompact for each $i = 1, \ldots, 2^n$.

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Proof. (A) implies (B). Take \( \{a_n X | n \in N \} \) as in (A). For each \( n \in N \), let \( f_n \) be the natural mapping of \( BX - X \) onto \( a_n X - X \) and where \( f_n \) is the identity on \( X \) (cf. [4]). Set \( U_i^n = f_n^{-1}(a^n_i) \), \( i = 1, \ldots, 2^n \), \( V_n = \bigcup \{U_i^n | 1 < i < 2^n; i \text{ odd} \} \), \( W_n = \bigcup \{U_i^n | 1 < i < 2^n; i \text{ even} \} \), for each \( n \in N \). Then \( V_n \) and \( W_n \) partition \( \beta X - X \) into disjoint, nonempty open sets. Let \( X_n = X \cup \{b_n, c_n\} \) be the two-point compactification of \( X \) determined by identifying \( V_n \) and \( W_n \) to points \( b_n \) and \( c_n \), respectively. Let \( p_n \) be the natural mapping of \( \beta X \) onto \( X_n \), for each \( n \in N \). Take \( Y = X \times n \in N X_n \) and denote points of \( Y \) by \( (y_n) \), where \( y_n \in X_n \). Embed \( X \) in \( Y \) by letting \( \varphi(x) = (y_n) \), where \( y_n = x \), for all \( n \in N \). Suppose \( (d_n) \in Y \) satisfies \( d_n = b_n \) or \( d_n = c_n \), for each \( n \in N \). Our aim is to show that \( \text{Cl}_Y (\varphi(X) - \varphi(X)) \) consists of precisely such points. To this end, let \( \pi_n \) be the projection of \( Y \) onto \( X_n \) and suppose that \( G = \pi_n^{-1}(G_n) \cap \cdots \cap \pi_1^{-1}(G_1) \) is a basic neighborhood of \( (d_n) \) in \( Y \). If \( d_n = b_n \), then \( p_n^{-1}(G_n) \) contains \( V_n \) and if \( d_n = c_n \), then \( W_n \subseteq p_n^{-1}(G_n) \). Now, for \( n > 1 \), \( t_{n+1} \circ f_{n+1} = f_n \), so that for \( i = 1, 2, \ldots, 2^n+1 \), \( U_i^{n+1} \subseteq U_i^{n} \), where \( \lceil \rceil \) is the greatest integer function. It follows that some \( U_i^{n+1} \subseteq p_n^{-1}(G_n) \), for all \( i = 1, \ldots, k \). Since \( U_i^n \) is nonempty and \( X \) is dense in \( \beta X \), we can select \( w \in X \) such that \( w \in p_n^{-1}(G_n) \cap \cdots \cap p_1^{-1}(G_1) \). Then \( \varphi(w) \in G \cap \varphi(X) \), so that \( (d_n) \in \text{Cl}_Y (\varphi(X) - \varphi(X)) \).

Next, suppose \( (d_n) \in Y \), where each \( d_n \in X \) but \( d_i \neq d_j \), for some \( i \neq j \). Then there exist disjoint neighborhoods \( U \) and \( V \) in \( X \) of \( d_i \) and \( d_j \), respectively. Now \( \pi_i^{-1}(U) \cap \pi_j^{-1}(V) \) is a neighborhood of \( (d_n) \) which contains no point of \( \varphi(X) \). Similarly, if, for some \( i \neq j \), \( d_i \in X \) and \( d_j \in Y \), then \( (d_n) \notin \text{Cl}_Y \varphi(X) \). Thus, \( \text{Cl}_Y \varphi(X) \) is a compactification of \( X \) whose remainder is a homeomorph of the (usual) Cantor set \( \mathcal{C} \). It follows from a theorem of Magill [9] that any compact metric space is a remainder of \( X \).

(B) implies (C). Let \( \alpha X \) be a compactification of \( X \) with remainder \( \mathcal{C} \). For each \( n \in N \), let \( \{a^n_i | i = 1, \ldots, 2^n \} \) be a collection of closed subsets of \( \mathcal{C} \) such that \( \mathcal{C} = \bigcup \{a^n_i | i = 1, \ldots, 2^n \} \) and \( A_{2^n+1} \cup A_{2^n+1} = a^n_i \), \( i = 1, \ldots, 2^n \). In \( \alpha X \) choose disjoint open sets \( H_i \) with \( a^n_i \subseteq H_i \), for \( i = 1, 2 \). Set \( G_i = H_i \cap X \), \( i = 1, 2 \). Clearly \( K_1 = X - (G_1 \cup G_2) \) is compact but \( K_1 \cup G_1 \) and \( K_1 \cup G_2 \) are noncompact. Proceeding inductively, assume that a collection \( \mathcal{G}_n \) has been defined as in (C). Select a pairwise disjoint family \( \{H_i^{n+1} | i = 1, \ldots, 2^{n+1} \} \) of open sets in \( \alpha X \), such that \( A_{2^{n+1}} \subseteq H_{i+1}^{n+1} \), \( i = 1, \ldots, 2^{n+1} \), and set \( G_{2^{n+1}} = H_{2^{n+1}} \cap G_i^{n} \) and \( G_{2^{n+1}} = H_{2^{n+1}} \cap G_i^{n} \). This defines \( \mathcal{G}_{n+1} \) according to (C). Hence the sequence \( \{\mathcal{G}_n | n \in N \} \) satisfies (C).

(C) implies (A). We utilize Magill's construction in [7] to obtain a sequence of compactifications \( \{\alpha_n X | n \in N \} \) subject to (A). Accordingly, let \( \alpha_n X - X = \{a^n_i | i = 1, \ldots, 2^n \} \), where basic neighborhoods of each \( a^n_i \) are sets \( \emptyset \cup \{a^n_i \} \), for \( \emptyset \) open in \( X \) and \( (K_n \cup G_i^n) - \emptyset \) compact.

Define mappings \( t_{n+1} \) from \( \alpha_{n+1} X \) onto \( \alpha_n X \) as in (A). Evidently, each \( t_{n+1} \) is continuous at points of \( X \) since \( X \) is locally compact. Next, let \( \emptyset \cup \{a^n_i \} \) be any basic neighborhood of \( a^n_i \). Consider \( \emptyset \cup \{a^n_{i+1} \} \). Since \( (K_{n+1} \cup G_{2^{n+1}}) - \emptyset \) is a closed subset of a compact set. Thus \( (K_{n+1} \cup G_{2^{n+1}}) - \emptyset \) is compact and \( \emptyset \cup \{a^n_{i+1} \} \) is a neighborhood
of $a_{2i}^{n+1}$ in $\alpha_{n+1}X$. It now follows that $t_{n+1}$ is continuous at $a_{2i}^{n+1}$. In this manner $t_{n+1}$ is demonstrated to be continuous at each point of $\alpha_{n+1}X$ and the sequence \( \{\alpha_nX|n \in N\} \) satisfies (A). This complete the proof.

3. Sufficiency conditions and examples. The following is immediate from Theorem 2.1.

**Corollary 3.1.** (A) If $X$ contains a family \( \{G_n|n \in N\} \) of pairwise disjoint open sets such that $K = X - \bigcup \{G_n|n \in N\}$ is compact and all $K \cup G_n$ are noncompact, then all compact metric spaces are remainders of $X$.

(B) If $X$ is the (topological) free union of a compact space and an infinite discrete space, then all compact metric spaces are remainders of $X$.

The converse of 3.1(A) is false. For, if $X$ is the closed unit square with $C \times \{0\}$ deleted, then all compact metric spaces are remainders of $X$, but $X$ contains no family of open sets satisfying the requirements of 3.1(A).

Evidently, if $X$ satisfies (A)–(C) of Theorem 2.1, then $X$ has a countable remainder. The converse is false. If $X = W \times W^*$, where $W$ is the space of all countable ordinals, then $\beta X - X = W^*$ (cf. 8L and 8M of [4]). Since any compact metric space which is a continuous image of $W^*$ must be countable or finite, it follows from Magill's theorem [9] that (B) of 2.1 cannot hold for $X$. However, $X$ has a compactification with countable remainder since $\beta X - X$ has infinitely many components (cf. [8]).

**References**


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