PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 3, July 1981

ON STOKES' THEOREM FOR NONCOMPACT MANIFOLDS

LEON KARP[1]

ABSTRACT. Stokes' theorem was first extended to noncompact manifolds by Gaffney. This paper presents a version of this theorem that includes Gaffney's result (and neither covers nor is covered by Yau's extension of Gaffney's theorem). Some applications of the main result to the study of subharmonic functions on noncompact manifolds are also given.

0. In [**4**] Gaffney extended Stokes' theorem to complete Riemannian manifolds M^n and proved that if $\omega \in \Lambda^{n-1}(M^n)$ and $d\omega$ are both integrable then $\int_M d\omega = 0$. The same conclusion was established by Yau under the sole condition that $\liminf_{r\to\infty} r^{-1} \int_{B(r)}|\omega|\, d\,\mathrm{vol} = 0$, where $B(r)$ = geodesic ball of radius r about some point $p \in M$ (cf. the remarks in the appendix to [**13**]). The purpose of this note is to give another extension of Gaffney's theorem that covers some cases not included in Yau's result.

1. Before formulating our main theorem we recall some background material. Let M^n be a Riemannian n-manifold and suppose that, in local coordinates $(x^1, \ldots, x^n)$, the vector field X is represented as $X = \sum X^i\, \partial/\partial x^i$, the metric tensor is given as (g_{ij}), and $g = \det(g_{ij})$. The quantities given locally as $\sqrt{g}$ and $\sum(\partial/\partial x^i)(\sqrt{g}X^i)$ are then *densities* (i.e., under a coordinate change they are multiplied by the absolute value of the Jacobian corresponding to the change of coordinates), and hence they may be unambiguously integrated via local coordinates even if M^n is not orientable ([**10**], cf. pp. 21–26 of [**9**] where a different terminology is used). When f is a scalar (i.e., has a local expression that is invariant under change of coordinates), we will denote the integral of the density $f\sqrt{g}$ by $\int f\, dv_g$.

If M^n is an oriented Riemannian manifold, then $\sqrt{g}\,dx^1 \wedge \cdots \wedge dx^n$ represents the volume form v_g in local coordinates, and for the scalar $(1/\sqrt{g})\sum(\partial/\partial x^i)(\sqrt{g}X^i)$, called the *divergence* of X and denoted $\mathrm{div}_g X$, we have $\int_M \mathrm{div}_g X \cdot v_g = \int_M \mathrm{div}_g X \cdot dv_g$. Here the integrand on the left is an n-form, and the integrand on the right is actually the density $\sqrt{g} \cdot \mathrm{div}_g X$.

On any oriented manifold with volume form $\omega > 0$, one defines the divergence of X with respect to ω, denoted $\mathrm{div}_\omega X$, via $d(i_X\omega) = \mathrm{div}_\omega X \cdot \omega$, where i_X denotes contraction with X (cf. [**8**]), and it is easily seen that $\mathrm{div}_g X = \mathrm{div}_\omega X$ if $\omega = v_g$ for

Received by the editors September 5, 1980.
1980 *Mathematics Subject Classification*. Primary 58G99, 53C99; Secondary 53C20, 58C99.
Key words and phrases. Stokes' theorem, divergence theorem, subharmonic function, harmonic function, Gaussian curvature, sectional curvature.
[1]Research partially supported by NSF Grant MCS 76-23465.

some metric g. Furthermore, if α is an $(n-1)$-form on an orientable n-manifold, one can always write $\alpha = i_X v_g$ where g is an arbitrary metric, and X depends on g and α. (In fact, $X =$ the vector field dual to the 1-form $*_g\, \alpha$, where $*_g$ is the Hodge star operator relative to g (cf. [**10**, Chapter 1]).) Thus when α is an $(n-1)$-form with compact support in (the interior of) an orientable manifold, the fact that $\int_M d\alpha = 0$ (Stokes' theorem) follows from the more general result that

$$\int_M \operatorname{div}_g X \cdot dv_g = 0 \tag{$*$}$$

if X has compact support in (the interior of) a (not necessarily oriented) Riemannian manifold. A simple proof of $(*)$ may be culled from the argument on p. 26 of [**9**].

We formulate our extension of Stokes' theorem in the general framework of integration of densities, as in $(*)$, and we set (as usual): $f^+ = \max(0, f)$, $f^- = \max(0, -f)$, and (for a fixed metric g) $\operatorname{div} X = \operatorname{div}_g X$, $B(r) =$ the geodesic ball of radius r and center at some fixed $p \in M^n$, and $\|X\| =$ the length of the vector field X. If $u \in C^2$, then $\Delta u =_{\text{def}} \operatorname{div}(\operatorname{grad} u)$ where $\operatorname{grad} u$ is the unique vector field that satisfies $g(X, \operatorname{grad} u) = du(X)$ for all vector fields X. Recall that u is *subharmonic* (resp. *harmonic*) if $\Delta u \geqslant 0$ (resp. $\Delta u = 0$).

THEOREM. *Let M^n be a complete noncompact Riemannian n-manifold and X a vector field such that*

$$\liminf_{r \to \infty} \frac{1}{r} \int_{B(2r)/B(r)} \|X\|\, dv = 0.$$

If $\operatorname{div} X$ *has an integral (i.e., if either* $(\operatorname{div} X)^+$ *or* $(\operatorname{div} X)^-$ *is integrable) then* $\int_M \operatorname{div} X\, dv = 0$. *In particular, if outside some compact set* $\operatorname{div} X$ *is everywhere* > 0 (*or* < 0) *then* $\int_M \operatorname{div} X\, dv = 0$.

PROOF. Without loss of generality, we may assume that $(\operatorname{div} X)^-$ is integrable. It is known (cf. Andreotti-Vesentini [**1**] or Yau [**13**]) that there is a constant $C > 0$ such that for each $r > 0$ $\exists$ a Lipschitz continuous function φ satisfying: $0 \leqslant \varphi \leqslant 1$ on M, $\varphi \equiv 1$ on $B(r)$, $\varphi \equiv 0$ on the complement of $B(2r)$ and $\|\operatorname{grad} \varphi(x)\| \leqslant C/r$ on M. Integrating $\operatorname{div}(\varphi^2 X)$ over $B(2r)$ and applying the usual divergence theorem $(*)$ (which is valid in this situation, as is evident from [**9**, p. 26]), we find $0 = \int_{B(2r)} \operatorname{div}(\varphi^2 X) \cdot dv$ and so

$$\left| \int_{B(2r)} \varphi^2 \operatorname{div} X \cdot dv \right| \leqslant \frac{\text{const}}{r} \int_{B(2r)/B(r)} \|X\|\, dv.$$

Thus,

$$\int_{B(r)} (\operatorname{div} X)^+\, dv - \int_M (\operatorname{div} X)^- dv \leqslant \frac{\text{const}}{r} \int_{B(2r)/B(r)} \|X\|\, dv.$$

We may choose $r = r_i \to \infty$ such that the right-hand side of this inequality tends to zero. Consequently, $(\operatorname{div} X)^+$ is also integrable and $\int_M \operatorname{div} X\, dv \leqslant 0$. Since $(\operatorname{div} X)^+$ is now known to be integrable, the same argument may be repeated with $-X$ in place of X. Thus $\int_M \operatorname{div} X\, dv = 0$ and the proof is complete.

The theorem has the following consequences:

COROLLARY 1. *Let M^n be a complete noncompact Riemannian manifold of qth-order volume growth (i.e., $\exists\, c > 0$ and $q \geq 1$ such that* $\operatorname{vol}(B(r)) \leq Cr^q$ *for* $r \geq 1$). *If* $\operatorname{div} X \geq 0$ *outside of some compact set and either* (a) $q > 1$ *and* $X \in L^p(M, dv)$ *where* $1/p + 1/q = 1$ *or* (b) $q = 1$ *and* $\|X\| \to 0$ *uniformly at infinity in M, then* $\int_M \operatorname{div} X\, dv = 0$.

PROOF. It suffices to apply Hölder's inequality and the fact that $\int_{B(2r)/B(r)} \|X\|^p\, dv \to 0$ if $q > 1$ (i.e., $p < \infty$) and $\limsup \operatorname{vol}(B(2r)/B(r))/r < \infty$ if $q = 1$.

REMARK. Recall that if M has nonnegative Ricci curvature then M has nth-order volume growth (cf. Bishop and Crittenden [2]). This fact is used in the proof of

COROLLARY 2. *If M^2 is a complete noncompact Riemannian 2-manifold with nonnegative Gaussian curvature then it admits no nonconstant C^2 subharmonic function with finite Dirichlet integral* $\int_M \|\operatorname{grad} u\|^2\, dv$.

PROOF. It follows from Corollary 1 and the remark above that if $\Delta u \geq 0$ and $\operatorname{grad} u \in L^2$ then $\int \Delta u = 0$. Thus, any such function is harmonic and so a classical computation of Bochner [3] shows that $\|\operatorname{grad} u\|^2$ is subharmonic. Since the manifold has nonnegative curvature, a result of Greene-Wu [5] gives $\int_M \|\operatorname{grad} u\|^2\, dv = +\infty$ unless $u \equiv$ constant.

REMARKS. (1) As its proof shows, the result of Corollary 2 is known for harmonic functions. Moreover, for this case a proof can be given by combining the following two classical results (as remarked by Greene-Wu [5]): (a) If a Riemann surface admits a Dirichlet-finite harmonic function then it also admits a bounded Dirichlet-finite harmonic function (Virtanen [11]). (b) A Riemann surface with nonnegative Gaussian curvature admits no nonconstant bounded harmonic functions (Blanc-Fiala-Huber, cf. Huber [6] and Yau's generalization [12]).

(2) The result and proof of Corollary 2 are actually valid for n-manifolds in the following form:

COROLLARY 2.′. *If M^n is a complete, noncompact Riemannian n-manifold of nonnegative sectional curvature then no nonconstant C^2 subharmonic function u has* $\operatorname{grad} u \in L^{n/n-1}(M, dv)$.

In fact, weaker curvature conditions suffice but we omit the details.

(3) The assertions of Corollaries 1, 2, and 2′ are sharp. In fact, it is easy to construct C^2 functions u that are subharmonic but not harmonic on $\mathbf{R}^n$ and satisfy $0 \neq \|\operatorname{grad} u\| \leq C(1/r)^{n-1}$ in $r > 1$. These functions thus have the property that $\operatorname{grad} u \in L^{(n/n-1)+\varepsilon}$ for every $\varepsilon > 0$. An example of such a function is

$$u = \begin{cases} p(x), & |x| \leq 1 \\ -r^{2-n}, & |x| > 1 \end{cases} \quad \text{on } \mathbf{R}^n, n \geq 3,$$

where $p(x)$ is a polynomial of degree four whose coefficients are chosen so that $u \in C^2$ and $\Delta u \geq 0$ (and it is easy to check that this can be done).

(4) Related results appear in [13] and [7].

ACKNOWLEDGEMENT. We are indebted to the referee for suggesting the inclusion of an expanded summary of the background material in §1 in place of the briefer discussion in the original manuscript.

BIBLIOGRAPHY

1. A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. **5** (1965), 313–362.

2. R. Bishop and R. J. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.

3. S. Bochner, *Curvature and Betti numbers*, Ann. of Math. **49** (1948), 379–390.

4. M. Gaffney, *A special Stokes' Theorem for complete Riemannian manifolds*, Ann. of Math. **60** (1954), 140–145.

5. R. Greene and H. Wu, *Integrals of subharmonic functions on manifolds of nonnegative curvature*, Invent. Math. **27** (1974), 265–298.

6. A Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72.

7. L. Karp, *Subharmonic functions on real complex manifolds*, Math. Z. (to appear).

8. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*. I, Interscience, New York, 1961.

9. G. de Rham, *Variétés différentiables*, Hermann, Paris, 1955.

10. S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

11. K. I. Virtanen, *Über die Existenz von becshränkten harmonischen Funktionen auf offenen Riemannschen Flächen*, Ann. Acad. Sci. Fenn. Ser. A I Math.-Phys., no. 75, (1950), 8pp.

12. S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

13. ______, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), 659–670.

DEPARTMENT OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

Current address: Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109