ON THE WEIGHT AND PSEUDOWEIGHT OF LINEARLY ORDERED TOPOLOGICAL SPACES

KLAAS PIETER HART

Abstract. We derive a simple formula for the weight of a LOTS using the pseudoweight. As an application we give a very short proof of the nonorderability of the Sorgenfrey-line.

1. Definitions. Let $(X, \tau)$ be a $T_1$-space. A collection $\mathcal{U} \in \tau$ is called a $\psi$-base for $X$ [3] if

(i) $\mathcal{U}$ covers $X$, and

(ii) $\cap \{ U \mid x \in U \in \mathcal{U} \} = \{ x \}$, for all $x \in X$.

We put as usual $\psi w(X) = \min \{| \mathcal{U} | \mid \mathcal{U} \text{ is a } \psi\text{-base for } X \}$.

Recall that

$$c(X) = \sup \{| \mathcal{U} | \mid \mathcal{U} \text{ and } \mathcal{U} \text{ is disjoint} \},$$

and

$$w(X) = \min \{| \mathcal{B} | \mid \mathcal{B} \subset \tau \text{ and } \mathcal{B} \text{ is a base for } X \}.$$ 

2.

Theorem. If $X$ is a Linearly Ordered Topological Space (LOTS), then $w(X) = c(X) \cdot \psi w(X)$.

Proof. " \Rightarrow \" is obvious.

" \Leftarrow \" . Let $\mathcal{U} = \{ U_i \}_{i \in I}$ be a $\psi$-base for $X$ with $| \mathcal{U} | = \psi w(X)$. For each $i \in I$ let $\{ C_{ij} \}_{j \in J_i}$ be the collection of convex components of $U_i$. Put $\mathcal{B} = \{ C_{ij} \mid j \in J_i, i \in I \}$. Since $| J_i | < c(X)$ for all $i$, we see that $| \mathcal{B} | < c(X) \cdot \psi w(X)$. We claim that $\mathcal{B}$ is a subbase for $X$.

Indeed, take $x \in X$ and $(a, b) \ni x$. Since $\cap \{ B \mid x \in B \in \mathcal{B} \} = \{ x \}$, there exist $B_a, B_b \in \mathcal{B}$ such that $x \in B_a \ni a$ and $x \in B_b \ni b$. Then $x \in B_a \cap B_b \subset (a, b)$. 

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3. Examples. We shall show that our theorem cannot be improved.

3.1. The Sorgenfrey-line $S$ shows that we cannot replace “$X$ is a LOTS” by “$X$ is a GO-space”. Indeed $w(S) = 2^\omega$, $c(S) = \omega$ and $\psi w(S) = \omega$ [take all intervals with rational endpoints]. This shows once again that $S$ is not a LOTS. For other, more involved, proofs, see for example [1] and [4].

3.2. The largest “natural” invariant below $c(X)$ (in the case of LOTS) is $l(X)$, the Lindelöf number of $X$. We shall see that we cannot replace $c$ by $l$:

Let $A \subset \mathbb{R}$ be a subset of cardinality $2^\omega$ with the property that for any closed set $C \subset \mathbb{R}$ either $C \subset A$ or $C \subset \mathbb{R} \setminus A$ we have that $C$ is countable.

Build a Michael-line $M(A)$ by isolating every $a \in A$. The resulting space is Lindelöf [6]. Now let $X$ be the associated LOTS of the GO-space $M(A)$ [5].

$X = \langle (x, n) \in \mathbb{R} \times \mathbb{Z} | x \in \mathbb{R} \setminus A \Rightarrow n = 0 \rangle$ endowed with the lexicographic order. The map $f: X \to M(A)$ defined by $f(\langle x, n \rangle) = x$ is a retraction with countable fibers; hence $X$ is Lindelöf.

Let us put $U_q = \{x \in X | x < \langle q, 0 \rangle\}$, $V_q = \{x \in X | x > \langle q, 0 \rangle\}$ and $O_n = \{\langle a, n \rangle | a \in A\}$. Then $\mathcal{U} = \{U_q\}_{q \in \mathbb{Q}} \cup \{V_q\}_{q \in \mathbb{Q}} \cup \{O_n\}_{n \in \mathbb{Z}}$ is a countable $\psi$-base for $X$. Finally we have $w(X) = |A| = 2^\omega$.

3.3. The largest invariant—to our knowledge—below $\psi w(X)$ is $p_{sw}(X) = \min\{\text{ord}(\mathcal{U}) | \mathcal{U} \text{ is a } \psi\text{-base for } X\}$ [2]. If we let $Z$ be a dense left-separated subspace of a connected Souslin-line, then $Z$ is a LOTS with a point-countable base [7]. Thus we have $w(Z) = \omega_1 > \omega = c(Z) \cdot p_{sw}(Z)$; hence $\psi w$ cannot be replaced by $p_{sw}$.

REFERENCES


SUBFACULTEIT WISKUNDE, VRIJE UNIVERSITEIT, DE BOOLELAAN 1081, AMSTERDAM, THE NETHERLANDS