A RESULT RELATED TO A THEOREM BY PIANIGIANI

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Abstract. Let \( r: J \to J \) be a piecewise \( C^2 \) map, where \( J \) is an interval, satisfying \( \inf |r'| > 1 \). An upper bound for the number of independent absolutely continuous measures invariant under \( r \) is presented.

Introduction. Let \( J = [a, b] \) be an interval, \( \mathcal{B} \) the Lebesgue measurable subsets of \( J \), and \( \lambda \) the Lebesgue measure on \( J \). Let \( r: J \to J \) be a piecewise \( C^2 \) transformation satisfying \( \inf |r'(x)| > 1 \) where the derivative exists. In [1] it is shown that \( r \) admits an absolutely continuous invariant measure \( \mu \), i.e., \( \mu(A) = \mu(r^{-1}(A)) \) for all \( A \in \mathcal{B} \), and

\[
\mu(A) = \int_A f \, d\lambda,
\]

where we refer to \( f \) as the density invariant under \( r \). Clearly \( f > 0 \) and \( f \in L_1 \), the space of integrable functions on \( J \).

Let \( \mathcal{F}_r \) denote the space of densities invariant under \( r \) and \( \{a_1, a_2, \ldots, a_k\} \) those points in \( J \) where \( r' \) does not exist. The main result of [2] asserts that \( \dim \mathcal{F}_r < k \).

In fact it is very easy to establish a better bound. Let \( a = b_0 < b_1 < \ldots < b_m < b_{m+1} = b \) be the partition of \( J \) such that \( r \) is continuous and monotonic on each interval \((b_{j-1}, b_j)\). Clearly \( m < k \), and \( \dim \mathcal{F}_r < m \). In the special case where \( r \) is continuous on \( J \), the total number of peaks and valleys in the graph of \( r \) constitutes an upper bound for \( \dim \mathcal{F}_r \).

In §3 of [3] a still better bound is established for \( \dim \mathcal{F}_r \). Let \( \{b_1, b_2, \ldots, b_m\} \) be the partition defined in the previous paragraph. For each \( 1 < j < m \), define the pair

\[
\langle u_j, v_j \rangle = \langle r(b_j^-), r(b_j^+) \rangle,
\]

where \( u_j \) is regarded as \( u_j^+ \) or \( u_j^- \) depending on whether \( r(a_j - \epsilon) > u_j \) or \( r(a_j - \epsilon) < u_j \).

Two pairs \( \langle u_i, v_i \rangle \) and \( \langle u_j, v_j \rangle \) are said to be dependent if they have one or both coordinates in common. Otherwise the pairs are independent. Let \( \mathcal{N}_r \) denote the maximal number of independent pairs. Then Theorem 2 of [3] asserts that \( \dim \mathcal{F}_r < \mathcal{N}_r \). In this note we suggest a modified definition of dependence and present a different bound for the number of absolutely continuous measures invariant under \( r \).

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2. Dependence of densities. Let \( \tau : J \to J \) be piecewise \( C^2 \) satisfying \( \inf|\tau'(x)| > 1 \) and let \( \mathcal{D} = \{b_1, b_2, \ldots, b_m\} \) be the partition on which \( \tau \) is piecewise continuous and monotonic. We shall say that \( b_i \) and \( b_j \) are dependent if
\[
\tau(b_i - \epsilon, b_i + \epsilon) \cap \tau(b_j - \epsilon, b_j + \epsilon)
\]
has positive measure for every \( \epsilon > 0 \). This implies, but is not equivalent to
\[
\langle \tau(b_i^-), \tau(b_i^+) \rangle \cap \langle \tau(b_j^-), \tau(b_j^+) \rangle \neq \emptyset.
\]
This definition of dependence for a pair of discontinuities in \( \mathcal{D} \) is reflexive, symmetric, but not transitive. A collection \( S \subset \mathcal{D} \) is said to be dependent if every pair of points in this collection is dependent, and maximal if \( S \) is not a proper subset of any dependent collection. Notice that two distinct maximal dependent collections may have nonempty intersection, and such a collection may consist of a single point. Thus, given \( b_j \in \mathcal{D} \), there exists at least one and at most two maximal dependent collections containing \( b_j \). In particular, when \( \tau \) is continuous at \( b_j \), there exists only one maximal dependent collection containing this point. Let \( H_{\tau} \) be the number of distinct maximal dependent collections. Then, we have

**Theorem.** \( \dim \mathcal{F}_{\tau} < H_{\tau} \).

**Proof.** We first show that if \( f_1 \) and \( f_2 \) are invariant with disjoint supports, then to each \( f_i \) there corresponds one maximal dependent collection \( S_i \) and \( S_1 \neq S_2 \). Letting \( M_i = \text{spt } f_i \), it is easy to see that int \( M_i \) has to contain at least one point of \( \mathcal{D} \), say \( b_i' \). Let \( S_1 \) and \( S_2 \) be any maximal collections containing \( b_1' \) and \( b_2' \), respectively, and suppose \( S_1 = S_2 \). Then \( b_1' \) and \( b_2' \) are dependent. Since \( \tau(M_i) \subset M_i \) a.e. [1], and \( (b_i' - \epsilon, b_i' + \epsilon) \subset M_i \) for some \( \epsilon < 0 \), the dependence of \( b_1' \) and \( b_2' \) implies
\[
\lambda(M_1 \cap M_2) > \lambda[\tau(b_1' - \epsilon, b_1' + \epsilon) \cap \tau(b_2' - \epsilon, b_2' + \epsilon)] > 0.
\]
This is a contradiction. Therefore, \( S_1 \) and \( S_2 \) must be distinct.

Now let \( \{f_1, f_2, \ldots, f_n\} \) be a maximal set of disjoint densities invariant under \( \tau \) [2]. By the preceding argument we see that there exists a 1-1 mapping from \( \{f_1, \ldots, f_n\} \) into \( \{S_1, \ldots, S_{H_{\tau}}\} \). Thus \( n < H_{\tau} \). Q.E.D.

3. Examples. (a) Consider the transformation \( \tau \) shown in Figure 1.

![Figure 1](http://www.ams.org/journal-terms-of-use)
(b) Let \( \tau \) have the graph shown in Figure 2.

For each discontinuity, we give the corresponding maximal dependent collection or collections as the case may be:

- \( b_1 \): \{\( b_1, b_3, b_5 \) and \( b_1, b_4 \),
- \( b_2 \): \{\( b_2, b_4, b_5 \),
- \( b_3 \): \{\( b_1, b_3, b_5 \),
- \( b_4 \): \{\( b_1, b_4 \) and \( b_2, b_4, b_5 \),
- \( b_5 \): \{\( b_1, b_3, b_5 \) and \( b_2, b_4, b_3 \),
- \( b_6 \): \{\( b_6 \).

There are 4 independent collections. Therefore \( \tau \) admits at most four independent invariant densities.

Notice that for this example the bound of [2] is 7, since there are 7 discontinuities of \( \tau' \) in \((0, 1)\).

REFERENCES


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