EXTENSIONS OF PURE POSITIVE FUNCTIONALS ON BANACH *-ALGEBRAS

R. S. DORAN AND WAYNE TILLER

Abstract. A known extension theorem for pure states on a Banach *-algebra with isometric involution is shown to hold for the wider class of Banach *-algebras with arbitrary, possibly discontinuous, involutions.

Let $A$ be a Banach *-algebra with isometric involution and bounded approximate identity $(e_n)$, and $B$ a closed *-subalgebra of $A$ containing $(e_n)$. In [3] G. Maltese proved that if $f$ is a pure state on $B$, then $f$ admits a pure state extension to $A$ if and only if $f$ admits a positive linear extension to $A$. Our purpose here is to extend this result to Banach *-algebras with arbitrary, possibly discontinuous, involutions.

For basic definitions and results from the theory of Banach *-algebras and their representations see [1], [2], or [4].

The following lemma handles the case when the algebra $A$ contains an identity.

**Lemma 1.** Let $A$ be a unital Banach *-algebra, $B$ a closed *-subalgebra of $A$ containing the identity $e$, and suppose that $f$ is a pure positive linear functional on $B$. Then $f$ can be extended to a pure positive linear functional on $A$ if and only if $f$ has a positive linear extension to $A$.

**Proof.** We may assume without loss of generality that $f(e) = 1$. Indeed, if $\lambda > 0$, then $\lambda f$ is pure and positive if $f$ is pure and positive. Our proof will be given in two steps:

I. $A$ has continuous involution;

II. $A$ has arbitrary involution.

**Proof of I.** Let $P_A$ denote the set of positive functionals $g$ on $A$ satisfying $g(e) = 1$. Define $P_B$ similarly. It is well known that a functional in $P_A$ (or $P_B$) is pure (pure on $B$) if and only if it is an extreme point of $P_A$ ($P_B$). Suppose, now, that $f$ has a positive linear extension to $A$, and set $X = \{ g \in P_A : g|_B = f \}$; i.e., $X$ is the set of all positive extensions of $f$. Then $X$ is nonempty by assumption, and it is clearly convex. We show that $X$ is compact in the relative weak *-topology. By the Banach-Alaoglu theorem it suffices to show that $X$ is weak *-closed and norm bounded. Suppose that $(g_n)$ is a net in $X$ and that $g_n \to g$. Then by the definition of the weak *-topology, $g_n(x) \to g(x)$ for every $x \in A$; thus, if $x \in B$, then

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\[ g(x) = \lim a_i g_i(x) = \lim a_i f(x) = f(x) \] which implies \( g \in X \). Therefore, \( X \) is weak *-closed.

Now let \( g \in X \) be arbitrary; since \( x \to x^* \) is continuous there exists \( k > 0 \) such that \( \|x^*\| < k\|x\| \) for all \( x \) in \( A \). Then, by [4, pp. 214, 219],

\[ |g(x)|^2 \leq g(e)g(x^*x) = g(e)A(x^*x) \leq \|x^*x\|^2 g(e) = k\|x\|^2, \]

where \( A(\cdot) \) denotes the spectral radius. Hence \( |g| < \sqrt{k} \) and \( X \) is norm bounded.

The Krein-Milman theorem now implies that \( X \) has extreme points. We denote the set of extreme points of \( X \), \( P_A \), and \( P_B \) by \( E(X) \), \( E(P_A) \), and \( E(P_B) \) respectively. Verification of the equality \( E(X) = X \cap E(P_A) \) will complete the proof of Part I.

Our proof follows that given in [3, p. 503].

It is clear that \( X \cap E(P_A) \subseteq E(X) \). Let \( g \in E(X) \) and suppose \( g = \frac{1}{2}(\phi + \psi) \), where \( \phi \), \( \psi \in P_A \). Then, taking restrictions, we obtain \( f = \frac{1}{2}(\phi|_B + \psi|_B) \). But \( \phi|_B \) and \( \psi|_B \) are in \( P_B \), and since \( f \in E(P_B) \), it follows that \( f = \phi|_B = \psi|_B \) which implies that \( \phi \) and \( \psi \) are in \( X \). Since \( g \) is an extreme point of \( X \) we have \( g = \phi = \psi \) which means that \( g \in E(P_A) \). Hence \( E(X) \subseteq X \cap E(P_A) \).

**Proof of II.** We now allow the involution to be arbitrary. If \( J \) denotes the Jacobson radical of \( A \), then \( A/J \) is a semisimple Banach *-algebra which, by Johnson’s uniqueness of the norm theorem [1, p. 130], has continuous involution. Hence the closure \( ((B + J)/J)^{-} \) in \( A/J \) of the *-subalgebra \( (B + J)/J \) is a Banach *-subalgebra of \( A/J \) containing the identity \( e + J \).

Let \( f' \) be the positive extension of \( f \) to \( A \), and define a function \( \tilde{f} : A/J \to \mathbb{C} \) by \( \tilde{f}(x + J) = f'(x) \). We note that \( \tilde{f} \) is well defined since \( f' \) is representable [4, p. 216] and \( J \) is contained in the reducing ideal. Furthermore, \( \tilde{f} \) is linear and positive and is therefore continuous since \( A/J \) has an identity. Moreover, if \( b \in B \), then \( \tilde{f}(b + J) = f'(b) = f(b) \). Let

\[ \tilde{f} = \tilde{f}'|_{((B + J)/J)^{-}}. \]

Then \( \tilde{f} \) is a continuous positive linear functional on \( ((B + J)/J)^{-} \) and \( \tilde{f}(b + J) = f(b) \) for every \( b \in B \). We assert that \( \tilde{f} \) is pure. Indeed, let \( \tilde{g} \) be an arbitrary positive functional on \( ((B + J)/J)^{-} \) satisfying \( \tilde{g} \leq \tilde{f} \). Then \( \tilde{g}(b*b + J) \leq \tilde{f}(b*b + J) = f(b*b) \) for every \( b \in B \). Define a positive functional \( g \) on \( B \) by \( g(b) = \tilde{g}(b + J) \). Clearly \( g \leq f \), and since \( f \) is pure, it follows that \( g = \lambda f \), where \( 0 < \lambda < 1 \). Hence, \( \tilde{g} = \lambda \tilde{f} \) on \( (B + J)/J \). But \( \tilde{g} \) and \( \tilde{f} \) are both continuous, and thus it follows that \( \tilde{g} = \lambda \tilde{f} \) on all of \( ((B + J)/J)^{-} \); therefore, \( \tilde{f} \) is pure.

By part I, \( \tilde{f} \) has a pure positive extension to \( A/J \) which we denote by \( h \). Define \( h' : A \to \mathbb{C} \) by \( h'(x) = h(x + J) \). Then \( h' \) is a positive functional on \( A \) and if \( b \in B \), then \( h'(b) = h(b + J) = \tilde{f}(b + J) = f(b) \). It remains only to show that \( h' \) is pure. Let \( g' \) be a positive functional on \( A \) satisfying \( g' < h' \). Then \( g'(x^*x) < h'(x^*x) = h(x^*x + J) \). Define a functional \( g \) on \( A/J \) by \( g(x + J) = g'(x) \); \( g \) is well defined since \( g' \) is representable. Clearly \( g \) is positive and \( g < h \); but \( h \) is pure, so \( g = \lambda h \) which implies \( g' = \lambda h' \). Hence \( h' \) is pure and the proof is complete.

The next lemma is well known from Banach *-algebras with isometric involution (see [2, 2.2.10, p. 34]). We give a simple proof for the case of an arbitrary involution. In what follows we assume that all bounded approximate identities are bounded by one.
**Lemma 2.** Let $A$ be a Banach $*$-algebra with bounded approximate identity $\{e_a\}$, $\pi$ a nondegenerate $*$-representation of $A$ on a Hilbert space $H$, and let $I$ denote the identity operator on $H$. Then $\lim_a \pi(e_a) = I$, where the limit is in the strong operator topology.

**Proof.** For each $x \in A$ we have $||\pi(e_a) - \pi(x)|| < ||\pi|| \cdot ||e_a x - x|| \to 0$. Hence $||\pi(e_a)\pi(x)\xi - \pi(x)\xi|| \to 0$ for every $x \in A$ and every $\xi \in H$. Since $\pi$ is nondegenerate, the set $\pi(A)H$ is dense in $H$. Now let $\eta \in H$ be arbitrary, $\varepsilon > 0$, and set $M = \max(||\pi||, 1)$. Then there exists $\xi \in H$ and $x \in A$ such that $||\pi(x)\xi - \eta|| < \varepsilon/3M$ and there exists $\alpha_0$ such that $\alpha > \alpha_0$ implies $||\pi(e_a)\pi(x)\xi - \pi(x)\xi|| < \varepsilon/3$. Then

$$||\pi(e_a)\eta - \eta|| \leq ||\pi(e_a)\eta - \pi(e_a)\pi(x)\xi|| + ||\pi(e_a)\pi(x)\xi - \pi(x)\xi|| + ||\pi(x)\xi - \eta||$$

$$< ||\pi|| \cdot ||e_a|| \cdot ||\eta - \pi(x)\xi|| + \varepsilon/3 + \varepsilon/3M < \varepsilon$$

completing the proof.

**Theorem 3.** Let $A$ be a Banach $*$-algebra with bounded approximate identity $\{e_a\}$ and suppose $B$ is a closed $*$-subalgebra of $A$ containing $\{e_a\}$. Let $f$ be a pure positive linear functional on $B$ admitting a positive linear extension $f'$ to $A$. Then $f$ has a pure positive linear extension to $A$.

**Proof.** Since $f$ and $f'$ are representable, we can write $f(b) = (\pi(b)\xi | \xi)$ and $f'(x) = (\pi'(x)\xi | \xi')$ for all $b \in B$, $x \in A$, and suitable vectors $\xi$ and $\xi'$ in the respective spaces of $\pi$ and $\pi'$. Then, by Lemma 2, $||\xi|| = ||\xi'|| = \lim_a f(e_a) = \lim_a f'(e_a) = ||\xi||^2 = ||\xi'||^2 = (\xi | \xi')$. Let $A_e$ and $B_e$ denote the Banach $*$-algebras obtained from $A$ and $B$ respectively by adjoining identities. Define $*$-representations $\pi'$ and $\pi$ of $A_e$ and $B_e$ respectively by $\pi'((x, X)) = \pi'(x) + XI$ and $\pi((b, X)) = \pi(b) + XI$, where $I$ denotes the identity operator. Let $\tilde{f}'[(x, \lambda)] = (\pi'(x, \lambda)\xi | \xi')$ and $\tilde{f}[(b, \lambda)] = (\pi((b, \lambda))\xi | \xi)$. Then $\tilde{f}'$ and $\tilde{f}$ are positive functionals on $A_e$ and $B_e$ respectively and

$$\tilde{f}'[(b, \lambda)] = (\pi'[((b, \lambda))\xi | \xi']) = (\pi'(b)\xi | \xi') + \lambda(\xi | \xi')$$

$$= (\pi(b)\xi | \xi) + \lambda(\xi | \xi) = (\pi((b, \lambda))\xi | \xi) = \tilde{f}[(b, \lambda)]$$

for every $(b, \lambda) \in B_e$. Now $f$ pure implies that $\pi$ is irreducible [2, 2.5.4, p. 43]; hence $\pi'$ is irreducible and thus $\tilde{f}'$ is pure. By Lemma 1, $\tilde{f}'$ has a pure positive extension, say $g$, to $A_e$. Hence there exists an irreducible $*$-representation $\pi_g$ and a cyclic vector $\xi_g$ such that $g((x, \lambda)) = (\pi_g((x, \lambda))\xi_g | \xi_g)$. So $\pi_g|_{A_e}$ is also irreducible, and therefore the functional $g_A$ defined on $A$ by $g_A(x) = (\pi_g|_{A_e}(x)\xi_g | \xi_g)$ is a pure positive functional on $A$. Moreover, $g_A(b) = g((b, 0)) = \tilde{f}((b, 0)) = (\pi((b, 0))\xi | \xi) = (\pi(b)\xi | \xi) = f(b)$.
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DEPARTMENT OF MATHEMATICS, TEXAS CHRISTIAN UNIVERSITY, FORT WORTH, TEXAS 76129