MARKOV OPERATORS AND QUASI-STONIAN SPACES

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Abstract. Let \( X \) be a quasi-stonian space, and let \( T \) be a \( \sigma \)-additive Markov operator on \( C(X) \). Ando proved that if all \( T \)-invariant probabilities are \( \sigma \)-additive, then \( T \) is strongly ergodic (and the space of fixed points is finite-dimensional). We prove that if the set of \( \sigma \)-additive \( T \)-invariant probabilities is weak-* dense in the set of all \( T \)-invariant probabilities, then \( T \) is strongly ergodic. This result is easy in case \( X \) is hyperstonian. Our method of proof is to use an idea of Gordon to "hyperstonify" part of our quasi-stonian space.

1. Introduction. A compact space is quasi-stonian if any of the following equivalent conditions hold: (1) if \( A \) is an open \( F_\sigma \) set, then closure\(^c A\) is clopen (= closed and open), (2) \( X \) is totally disconnected, and the algebra of clopen sets is \( \sigma \)-complete, (3) \( C(X) \) is conditionally \( \sigma \)-complete. A Borel measure \( m \) on \( X \) is \( \sigma \)-additive if \( f_n \in C(X), f_1 > f_2 > \cdots > 0, \) and \( \bigwedge f_n = 0 \) imply \( \lim m(f_n) = 0 \). The operator \( T \) on \( C(X) \) is \( \sigma \)-additive if \( \bigwedge T f_n = 0 \) whenever \( f_n \) is as above [An, Section 2]. Let \( \Sigma(X) \) be the Banach space of \( \sigma \)-additive elements of the dual space \( C(X)^* \).

If \( T \) is a Markov operator on \( C(X) \), i.e., \( T > 0 \) and \( Te = e \), where \( e \) is the unit function, let \( F(T) = \{ f \in C(X): Tf = f \}, F(T^*) = \{ m \in C(X)^*: T^* m = m \} \) (where \( T^* \) is adjoint of \( T \)), and \( P(T^*) \) be the set of probability measures in \( F(T^*) \).

Write \( T_n = (1/n)(I + \cdots + T^{n-1}) \). \( T \) is called strongly ergodic if for each \( f \) in \( C(X) \), \( \lim T_n f \) exists in the Banach space \( C(X) \). Our main result is

1.1. Theorem. If \( X \) is quasi-stonian, \( T \) a \( \sigma \)-additive Markov operator on \( C(X) \), and \( P(T^*) \cap \Sigma(X) \) is weak-* dense in \( P(T^*) \), then \( T \) is strongly ergodic.

1.2. Corollary [An, Theorem 2]. If \( X \) is quasi-stonian and \( T \) a \( \sigma \)-additive Markov operator on \( C(X) \), the following are equivalent:

(a) \( F(T^*) \subset \Sigma(X) \),
(b) \( T \) is strongly ergodic, and \( \dim(F(T)) = \dim(F(T^*)) < \infty \).

We note that Sato has extended Ando's theorem to include certain non-Markov positive operators [Sa, Theorem 1]. Sato's theorem is not in any obvious way a corollary to 1.1.

§2 is devoted to some general ergodic theory, and §3 to the proofs of 1.1 and 1.2.

1.3. Some concepts. A compact space is stonian if any of the following equivalent conditions hold: (1) if \( A \) is open, then closure\(^c A\) is open, (2) \( X \) is totally

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disconnected, and the Boolean algebra of clopen sets is complete, (3) $C(X)$ is a conditionally complete lattice.

To define hyperstonian spaces, we need the concept of normal measure, i.e., a measure $m$ which satisfies any of the equivalent conditions: (1) $m$ is order-continuous, i.e., if $\{f_o\}$ is a directed family in $C(X)$ and $\mathbf{f} = \sup_{o} f_o$, then $\int \mathbf{f} dm = \lim_o \int f_o dm$, (2) if $F$ is a closed nowhere dense set, then $m(F) = 0$ [Sch, p. 149, exercise 24]. Let $\Sigma_0 = \Sigma_0(X)$ be the Banach space of all normal measures.

A compact $X$ is hyperstonian if any of the following equivalent conditions hold: (1) $C(X) = (\Sigma_0)^*$, i.e., $C(X)$ is the Banach space dual of $\Sigma_0$ (where $\Sigma_0$ has its norm topology), (2) $X$ is stonian and $\Sigma_0$ is weak-* dense in $C(X)^*$, (3) $X$ is stonian and the union of the support sets of measures in $\Sigma_0$ is dense in $X$ [Sch, pp. 121–122].

Clearly, a normal measure is $\sigma$-additive, as we have defined it above. In the course of proving Theorem 1.1, we shall introduce a new topology under which, as we shall prove, the $\sigma$-additive measures become normal measures.

2. Criteria for ergodicity. If $R$ is an operator on the Banach space $B$, let $R^* = \text{adjoint}(R)$, $F(R) = \text{fixed points of } R$, and $F(R^*) = \text{fixed points of } R^*$. Write $R_n = (1/n)(I + \cdots + R^{n-1})$. $R$ is strongly ergodic if $\lim n R_n x$ exists for each $x$ in $B$. If $R$ is a contraction, then $R$ is strongly ergodic iff $F(R)$ separates $F(R^*)$, i.e., $F(R^*) \cap F(R)^\perp = \{0\}$ [S2]. More generally, the separation criterion is valid if $R$ satisfies

$$\|R_n\| < M (n > 1), \quad \text{and} \quad \lim \inf n^{-1}\|R^n\| = 0$$

[LL, p. 123]. Note that if (1) holds for $R$, then it holds for $R^*$, $R^{**}$, etc.

2.1. Proposition. If $R$ satisfies (1), the following are equivalent:

(a) both $R$ and $R^*$ are strongly ergodic,

(b) $F(R^{**}) = \sigma(B^{**}, B^*)$-closure of $F(R)$,

(c) norm-closure($(I - R)^*(B^*)) = \text{weak-* closure}((I - R)^*(B^*))$.

Proof. First note that in general weak-* closure($(I - R)^*(B^*)) = F(R)^\perp$, and $\sigma(B^{**}, B^*)$-closure($F(R)$) $= F(R)^{\perp\perp}$.

(c) implies (b). In general, $F(R^{**}) = \{\text{norm-closure of } (I - R)^*(B^*)\}^{\perp}$, so if norm-closure($(I - R)^*(B^*)) = F(R)^\perp$, then $F(R^{**}) = F(R)^{\perp\perp}$.

(b) implies (c). If (c) fails, then there exists $m$ in $F(R)^\perp \setminus \text{norm-closure of } (I - R)^*(B^*)$. By Hahn-Banach there exists $F$ in $B^{**}$ with $F(m) = 1$ and $F(n - R^* n) = 0$ for all $n$ in $B^*$. Since $F$ is in $F(R^{**}) \setminus F(R)^{\perp\perp}$, (b) fails.

(c) implies (a). In general, norm-closure($(I - R)^*(B^*)) \cap F(R^*) = \{0\}$, so if (c) holds, then $F(R)^\perp \cap F(R^*) = \{0\}$, and hence $R$ is strongly ergodic, by the separation theorem. Hence there exists a projection $S$ on $F(R)$ with kernel closure($I - R^*(B)$), and then $S^*$ yields the decomposition $B^* = F(R^*) \oplus F(R)^\perp$.

But then (c) implies $B^* = F(R^*) \oplus \text{norm-closure}((I - R)^*(B^*))$, and this implies (by direct computation of averages) that $R^*$ is strongly ergodic.

(a) implies (c). If $R$ is strongly ergodic, then (as noted above) $B^* = F(R^*) \oplus F(R)^\perp$. If $R^*$ is strongly ergodic, then $B^* = F(R^*) \oplus \text{norm-closure}((I - R)^*(B^*))$. Since norm-closure($(I - R)^*(B^*)) \subset F(R)^\perp$, we have equality.
In the next result, $X$ will be a hyperstonian space, as defined in §1.3, with normal measures $\Sigma_0$. Note that if $R$ is a $\sigma$-additive Markov operator, then $R^\ast(\Sigma_0) \subset \Sigma_0$. Let $S = R^\ast|\Sigma_0$. Then $S$ is norm continuous on $\Sigma_0$, and $S^\ast = R$, $S^{**} = R^\ast$.

2.2. PROPOSITION. Let $R$ be a $\sigma$-additive Markov operator on $C(X)$, where $X$ is hyperstonian. Then $R$ is strongly ergodic iff

$$F(R^\ast) = \text{weak-* closure}(F(R^\ast) \cap \Sigma_0(X)).$$

PROOF. Necessity. If $R$ is strongly ergodic, then there exists a projection $Q$ such that $R_n f \to Q f$ ($f$ in $C(X)$). $Q$ is easily seen to be $\sigma$-additive, and hence $Q^\ast(\Sigma_0(X)) \subset \Sigma_0(X)$. $F(Q^\ast)$ is weak-* closed and $Q^\ast$ is weak-* continuous, so $F(R^\ast) = F(Q^\ast) = Q^\ast(C(X)) = \text{weak-* closure}(Q^\ast(\Sigma_0(X)))$. Since $Q^\ast(\Sigma_0(X)) \subset \Sigma_0(X)$ and $Q^\ast(\Sigma_0(X)) \subset F(R^\ast)$, we have $F(R^\ast) \subset \text{weak-* closure}(\Sigma_0(X) \cap F(R^\ast))$.

Sufficiency. Put $S = R^\ast|\Sigma_0(X)$. Then, since $R = S^\ast$, $R^\ast = S^{**}$, $C(X) = (\Sigma_0)^*$, and $C(X)^* = (\Sigma_0)^{**}$, the conclusion reads $F(S^{**}) = \sigma(\Sigma_0^*, \Sigma_0)$-closure of $F(S)$.

By "(b) implies (a)" of Proposition 2.1, it follows that $S$ and $S^\ast = R$ are strongly ergodic.

3. PROOFS.

3.1. Proof of Theorem 1.1. First we need some topological observations. A closed set $S$ is a $P$-set if the intersection of a countable number of neighborhoods of $S$ is again a neighborhood of $S$, or, equivalently, if $A$ is an open $F_0$ which is disjoint from $S$, then closure$(A)$ is disjoint from $S$. By [Li, Lemma 1], if $X$ is quasi-stonian and $m$ a $\sigma$-additive regular Borel probability, then sup$(m)$ is a $P$-set. (By sup$(m)$ we mean its support, i.e., the smallest closed set $S$ such that $m(S) = m(X) = 1).$ (To make the paper more self-contained, we sketch a proof. Let $m$ be $\sigma$-additive, $S = \text{sup}(m)$, and $A$ an open $F_0$ set with $A \cap S = \emptyset$. We must show closure$(A) \cap S = \emptyset$. Since $X$ is compact and totally disconnected, $A = \bigcup A_n$, where the $A_n$ are clopen sets. If $f_n$ is the characteristic function of $A_1 \cup \cdots \cup A_n$, then $0 = \int f_n dm$ for all $n$. If $f$ is the characteristic function of the clopen set closure$(A)$, then $f = \bigvee f_n$, whence $0 = \int f dm$, and so closure$(A) \cap S = \emptyset$. Thus the support of each $m$ in $P(T^*) \cap \Sigma(X)$ in the hypothesis of Theorem 1.1 is a $P$-set. Let $K = \text{closure}(\bigcup \{\text{sup}(m): m \in P(T^*)\})$. Since $P(T^*) \cap \Sigma(X)$ is weak-* dense in $P(T^*)$, we have

$$K = \text{closure}(\bigcup \{\text{sup}(m): m \in P(T^*) \cap \Sigma(X)\}).$$

Since $X$ is quasi-stonian and each set in the union is a $P$-set, it follows from [V, Theorem 2] that $K$ is also a $P$-set. (Again, we sketch a proof. Let $K = \text{closure}(\bigcup K_a)$, where the $K_a$ are $P$-sets, and $A$ an open $F_0$ with $A \cap K = \emptyset$. Suppose closure$(A) \cap K \neq \emptyset$. Since closure$(A)$ is open, it follows that closure$(A) \cap K_a \neq \emptyset$ for some index $a$. But $A \cap K_a = \emptyset$ and $K_a$ is a $P$-set, so we have a contradiction.)

Now $K$ is a $T$-invariant set, i.e., $f|K = 0$ implies $T f|K = 0$ [Si, Theorems 1.1, 1.3]. Hence $T$ induces in a natural way an operator $T_0$ on $C(K)$. Namely if $g$ is in
C(K) and x is in K, let \( \tilde{g} \) in C(X) be an extension of g, and let \( T_0 \tilde{g}(x) = Tg(x) = \int \tilde{g} \, d(T^* \delta_x) \). Since K is a P-set, \([A_2, \text{Proposition 3.5}]\) implies that if \( T_0 \) is strongly ergodic, then so is T. Thus it suffices to prove that the restricted operator \( T_0 \) is strongly ergodic.

Imitating a construction of Gordon [G], let

\[ Y = \bigcup \{ \text{sup}(m) : m \text{ in } P(T^* \cap \Sigma(X)) \}. \]

We denote the original topology of X (and hence of Y and closure(Y) = K) by the name \( \alpha \). We shall show later that if \( f \) is in C(X), then \( Tf|_Y \) converges uniformly on Y, hence on K = closure(Y).

Following [G, Section 5], we introduce a new topology \( \delta \) on Y, having as base of open sets all sets of the form \( X_m = \text{sup}(m) \), where m is any \( \sigma \)-additive measure whose support is contained in Y. We show that \( X_m \cap X_p = X_q \) for some q in \( \Sigma(X) \) as follows: since \( X_m \) is a P-set and \( X_p \) a support set, \( X_m \cap X_p \) is \( \alpha \)-clopen in \( X_p \). If \( f \) is the characteristic function of \( X_m \cap X_p \), put \( dq = fdp \) \([A_1, \text{Theorem 1}]\). Clearly q is \( \sigma \)-additive. Moreover since \( X_m \cap X_p \) is \( \alpha \)-open in \( X_p \) for each \( X_m \), \( \delta \) is a weaker topology on \( X_p \) than \( \alpha \). Since \( \delta \) is Hausdorff and \( (X_p, \alpha) \) is compact, \( (X_p, \delta) \) is compact and \( \alpha = \delta \) on \( X_p \). Thus, each \( X_p \) is open and compact in the \( \delta \)-topology.

(To show \( \delta \) is Hausdorff, let \( x \neq y \) in \( X_p \), and find A, B clopen and disjoint in \( (X_p, \alpha) \) such that \( x \) is in A and \( y \) is in B. Then \( \chi_A dp \) and \( \chi_B dp \) define members of \( \Sigma \) with disjoint supports containing \( x \) and \( y \) respectively.)

Let \( \tilde{Y} \) be the Stone-\( \check{C} \)-ech compactification of \( (Y, \delta) \). To show that \( \tilde{Y} \) is hyperstotional we depart from Gordon’s procedure [G, Section 6] and show (i) \( Y \), and hence \( \tilde{Y} \), is extremally disconnected, and (ii) the union of the supports of normal measures is dense in \( \tilde{Y} \) \([S\check{c}, \text{p. 121}]\).

For (i), we shall be using the \( \delta \)-topology except where otherwise mentioned. Let \( V \) be open and let \( x \) be in closure(\( V \)). We must show \( x \) is in interior(closure(\( V \))). Fix an \( X_m \) containing \( x \). Then \( x \) is in \( V \cap X_m \), which is an open subset of \( X_m \). If \( X_p \) is any basic neighborhood of \( x \), \( X_p \cap (V \cap X_m) \) contains \( x \) and is an open subset of \( V \cap X_m \), and hence \( x \) is interior to \( V \cap X_m \) in the subspace topology of \( X_m \). But on \( X_m \), \( \alpha = \delta \), and \( X_m \) is extremally disconnected for \( \alpha \) \([S\check{e}, \text{Theorem 2.2}]\), so \( x \) is interior to the closure in \( X_m \) of \( V \cap X_m \). Hence there exists a basic open set \( X_q \) with \( x \in X_q \subset X_m \) and \( X_q \subset X_m \)-closure(\( V \cap X_m \)) \( \subset Y \)-closure(\( V \)). Thus, \( x \) is interior to closure(\( V \)).

For (ii) we show that if \( m \) is in \( P(T^*) \cap \Sigma(X) \), then with respect to the \( \delta \)-topology on \( \tilde{Y} \) it becomes a normal measure. As noted in §1.3, it suffices to show that if \( F \) is a \( \delta \)-closed nowhere dense set, then \( m(F) = 0 \). Let \( S = X_m \), a \( \delta \)-clopen set. Since \( S \) is open, \( F \cap S \) is closed and nowhere dense in the relative \( \delta \)-topology of \( S \); and since the relative \( \alpha \) and \( \delta \)-topologies coincide on \( S \), \( F \cap S \) is closed and nowhere dense in the relative \( \alpha \)-topology of \( S \). We now show that there exists in \( X \) an \( \alpha \)-closed \( G_\delta \) set \( W \), \( \alpha \)-nowhere dense in \( X \), such that \( F \cap S \subset W \cap S \). Let \( V_n \) be \( \alpha \)-open sets with \( F \cap S \subset V_n \) and \( m(F) = m(F \cap S) = \lim m(V_n) \). Since \( F \cap S \) is \( \alpha \)-closed, hence \( \alpha \)-compact, we may assume each \( V_n \) is \( \alpha \)-clopen, and \( Z = \bigcap V_n \) is an \( \alpha \)-closed \( G_\delta \) set with \( F \cap S \subset Z \) and \( m(Z \setminus F) = 0 \). Now \( Z \cap S \) is
\(\alpha\)-nowhere dense in \(S\). (For otherwise there exists \(\alpha\)-clopen \(B\) with \(\emptyset \neq B \cap S \subset Z \cap S\). \(B \cap S \not\subset F \cap S\), since \(F \cap S\) is \(\alpha\)-nowhere dense in \(S\), so \((B \setminus F) \cap S \neq \emptyset\), whence \(m(B \setminus F) > 0\). But \((B \setminus F) \cap S \subset (Z \setminus F) \cap S\), so \(m(Z \setminus F) > 0\), a contradiction.) Since \(X\) is quasi-stonian, the set \(A = \text{interior}(Z)\) is clopen. Since \(Z \cap S\) is nowhere dense in \(S\), \(A \cap S = \emptyset\), so \(W = Z \setminus A\) is a nowhere dense \(\alpha\)-closed \(G_{\delta}\) with \(W \cap S = Z \cap S \not\subset F \cap S\).

Finally we show \(m(W) = 0\), and hence \(m(F) = 0\). But \(W\) has the form \(W = \cap A_n\), where the \(A_n\) are \(\alpha\)-clopen, \(A_1 \supset A_2 \supset \ldots\). If \(f_n\) is the characteristic function of \(A_n\), then since \(\text{interior}(W) = \emptyset\), we have \(\bigwedge f_n = 0\), and so \(m(W) < \lim f_n dm = 0\). (ii) is proved.

We now show that our original operator \(T\) induces an operator \(S\) on \(C(\tilde{Y})\). If \(m\) is in \(P(T^*) \cap \Sigma(Y)\), then \(f\) in \(\delta\)-continuous function on \(Y\) and \(x\) in \(X_m\), \(f|X_m\) is \(\alpha\)-continuous on \(X_m\), and we define \(Sf(x) = \int f d(T^*\delta_x)\). Then \(Sf\) is defined on all \(\tilde{Y}\) as the Stone-Čech extension.

We complete the proof by showing that \(S\) is uniformly ergodic, whence, if \(f\) is in \(C(K)\), then \(T_n f|K\) converges uniformly, and hence so does \(T_n f|K\). To apply Proposition 2.2, we must prove \(P(S^*) \cap \Sigma_0(Y)\) is weak-* dense in \(P(S^*)\). (Note that the \(\delta\)-topology introduces no new regular Borel measures on \(Y\). For if \(m\) is one whose support is contained in \(Y\), then since each \(X_p\) is a \(\delta\)-clopen set, \(m(X_p) > 0\) whenever \(\sup(m) \cap X_p \neq \emptyset\). Hence \(\text{support}(m)\) is contained in a countable collection of the \(X_p\) sets. The \(\alpha\)-closure of the union of this collection is contained in \(Y\), and is also an \(X_p\) set: if \(X_{p(i)}\) are the sets in question, just let \(p = \sum 2^{-i} p(i)\).)

Suppose \(P(S^*) \cap \Sigma_0(Y)\) is not weak-* dense in \(P(S^*)\), and let \(m_0\) be in \(P(S^*)\), but not in \(\text{close}(P(S^*) \cap \Sigma_0(Y))\). By separation [D, p. 22], there exists \(f\) in \(C(Y)\) such that \(p(f) < -1\) for all \(p\) in \(P(S^*) \cap \Sigma_0(Y)\), and \(m_0(f) > 0\). Now \(\{S_n: n > 1\}\) is a norm bounded subset in \(C(Y)\), and since \(C(\tilde{Y})\) is the dual of \(\Sigma_0(Y)\), it is precompact. Let \(g\) in \(C(Y)\) be a \(\sigma(\Sigma_0(Y), C(\tilde{Y}))\) cluster point, and let \(S_{n(i)}: i \in I\) be a subset such that \(\int S_{n(i)} dp \to \int g dp\) for all \(p\) in \(\Sigma_0(Y)\). Hence \(\int g dm_0 > 0\) and \(\int f g dp \to -1\) for all \(p\) in \(\Sigma_0(Y)\). Further, \(S g = g\). (Proof: \(\|S_{n(i)} g - S_{m(j)} g\| < n(i)^{-1}\|S_{m(j)} g - g\| < 2n(i)^{-1}\|g\| \to 0\), whence \(\int S g dp = \int g dp\) for all \(p\) in \(\Sigma_0(Y)\). Since, as noted above, \(\Sigma_0(\tilde{Y})\) is weak-* dense in \(C(\tilde{Y})^*\), we have \(S g = g\).) If \(p\) is an extreme point of \(P(S^*)\), then \(g\) is constant on \(\text{sup}(p)\) [S1, Theorem 1.11]. If, further, \(\sup(p) \subset X_m\) for some \(m\) in \(P(S^*) \cap \Sigma_0(Y)\), then since by hypothesis \(P(T^*) \cap \Sigma(Y)\) is dense in \(P(T^*)\) and the \(\alpha\)-topology on \(X_m\) is the same as the \(\delta\)-topology, it follows that \(p(g) < -1\) and hence \(g < -1\) on \(\text{sup}(p)\). By Krein-Milman, \(m\) is in the weak-* closed convex hull of the extreme points of \(P(S^*)\), and it is elementary to show that these extreme points may be assumed to have supports contained in \(\text{sup}(m) = X_m\). Hence

\[X_m = \text{closure}(\bigcup \{\sup(p): p \text{ extreme}, \sup(p) \subset X_m\}),\]

whence \(g < -1\) on \(X_m\). It follows that \(g < -1\) on \(Y\), and, by density, on \(\tilde{Y}\). But this contradicts \(m_0(g) > 0\). This proves \(P(S^*) \cap \Sigma_0(Y)\) is weak-* dense in \(P(S^*)\), and we are through.
3.2. Proof of the corollary.

(a) implies (b). If (a), then the theorem implies that $T$ is strongly ergodic, and we must show that $\dim F(T) < \infty$. Since by [S, Theorem 1.11], $\dim F(T) < \text{number of extreme points in } P(T^*)$, it suffices to prove that the latter set is finite. Suppose $m_1, m_2, \ldots$ are distinct extreme points, and let $S_i = \sup(m_i)$. Since $T$ is strongly ergodic, the sets $S_i$ are pairwise disjoint. (By [S, Theorem 2.7], invariant functions separate the $m_i$, and by [S, Theorem 1.11], an invariant function is constant on each $S_i$.) Since $P(T^*) \subset \Sigma(X)$, the $S_i$ are $P$-sets, and hence a routine induction gives a pairwise disjoint sequence of clopen sets $A_i$ with $S_i \subset A_i$. Define $f_i$ in $C(X)$ to be the characteristic function of $B_i = \text{closure } \bigcup_{n \geq 1} A_n$. Then $\bigwedge f_i = 0$, while if $m$ is a weak-* cluster point of $\{m_i\}$, then $\int f_i \ dm = 1$ for all $n$. Thus $m$ is an element of $P(T^*)$ which is not $\sigma$-additive, contrary to hypothesis.

(b) implies (a). This is the easy part; see [An].

Added in proof. In his thesis, D. Axmann gives a treatment of Ando’s condition in the context of vector lattices, and shows that under this condition $T$ is actually ergodic in the uniform operator topology (Struktur und Ergodentheorie irreduzibler operatoren auf Banachverbänden, Dissertation, Eberhard-Karls-Universität zu Tübingen, 1980, Chapter 4). I do not know whether our condition implies uniform ergodicity.

References


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