ADMISSIBLE KERNELS FOR STARSHAPED SETS

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Abstract. Steven Lay has posed the following interesting question: If \( D \) is a convex subset of \( \mathbb{R}^d \), then is there a starshaped set \( S \supset D \) in \( \mathbb{R}^d \) whose kernel is \( D \)? Thus the problem is that of characterizing those convex sets which are admissible as the kernel of some nonconvex starshaped set in \( \mathbb{R}^d \). Here Lay's problem is investigated for closed sets, and the following results are obtained: If \( D \) is a nonempty closed convex subset of \( \mathbb{R}^2 \), then \( D \) is the kernel of some planar set \( S \supset D \) if and only if \( D \) contains no line. If \( D \) is a compact convex set in \( \mathbb{R}^d \), then there is a compact set \( S \supset D \) in \( \mathbb{R}^d \) whose kernel is \( D \).

1. Introduction. Let \( S \) be a subset of \( \mathbb{R}^d \). Set \( S \) is said to be starshaped if and only if there is some point \( p \) in \( S \) such that \([p, x] \subseteq S\) for every \( x \) in \( S \). The set of all such points \( p \) is called the (convex) kernel of \( S \), denoted \( \ker S \). While many interesting results have been obtained for starshaped sets and their kernels [2], a new problem has been posed recently by Steven Lay [1]. If \( D \) is a convex subset of \( \mathbb{R}^d \), then is there a starshaped set \( S \supset D \) in \( \mathbb{R}^d \) whose kernel is \( D \)? That is, can we characterize those convex sets which are admissible as the kernel of some nonconvex starshaped set in \( \mathbb{R}^d \)? The question may be answered quickly in various special cases. In particular, when \( D \) is not a \((d - 1)\)-flat and \( D \) has dimension less than \( d \), it is not hard to construct an appropriate set \( S \). Similarly, when \( D \) is a polytope, such an \( S \) exists. However, the problem for arbitrary \( D \) is both intriguing and challenging. Here we begin by considering a closed convex set \( D \) in the plane, and constructions for \( S \) are given when \( D \) contains no line. If \( D \) is closed and contains a line, then it is proved that no such set \( S \) exists. Finally, extensions from \( \mathbb{R}^2 \) to \( \mathbb{R}^d \) are obtained with certain modifications.

The following definitions and terminology will be used throughout the paper. Let \( S \) be a subset of \( \mathbb{R}^d \). A point \( x \) in \( S \) is said to be a point of local convexity of \( S \) if and only if there is some neighborhood \( N \) of \( x \) such that \( N \cap S \) is convex. In case \( S \) fails to be locally convex at point \( q \) in \( S \), then \( q \) is called a point of local nonconvexity (lnc point) of \( S \). For \( S \) a 2-dimensional convex set, line \( L \) is said to support \( S \) if and only if \( L \) contains no interior point of \( S \) and the distance from \( L \) to \( S \) is zero. (Notice that such a line need not meet \( S \).) The terms convex, affine, closure, boundary, and interior of set \( S \), respectively. When \( S \) is convex, \( \dim S \) will be the dimension of set

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S. For distinct points x and y, dist(x, y) will represent the distance from x to y, 
L(x, y) will represent the line determined by x and y, and R(x, y) will be the ray 
eaming from x through y.

2. The results in \( \mathbb{R}^2 \). We begin with the following theorem for compact convex 
sets in \( \mathbb{R}^2 \).

**Theorem 1.** Let \( D \) be any nonempty compact convex set in \( \mathbb{R}^2 \). Then there is a 
compact set \( S \supseteq D \) in \( \mathbb{R}^2 \) with \( \text{ker} \, S = D \).

**Proof.** We begin by disposing of the cases for \( \dim D < 2 \). If \( D \) is a singleton set, 
let \( S \) be the union of two closed disks whose intersection is exactly \( D \). If \( D \) is a 
segment \([x, y]\), choose \( x_0, y_0 \) on \( L(x, y) \) with \( x_0 < x < y < y_0 \), choose \( z \notin L(x, y) \), 
and let \( S = \text{conv}(x, y, z) \cup [x_0, y_0] \).

Throughout the remainder of the argument, we assume that \( D \) is a full 2-dimen-
sional set. For convenience of notation, we suppose that \( D \) lies in the unit disk \( \{z: |z| = 1\} \), with the origin \( \theta \) interior to \( D \). Moreover, assume that \( D \) is oriented so 
that it has a horizontal line of support which intersects the set on the positive y 
axis. Let \( T_L \) and \( T_R \) denote the vertical lines which support \( D \); \( T_L \) in the left open 
halfplane, \( T_R \) in the right open halfplane. For each point \( x \notin D \), there are two 
distinct segments \((x, y_1)\) and \((x, y_2)\) disjoint from \( D \) whose lines support \( D \) at \( y_1 \) 
and \( y_2 \), respectively. We call \([x, y_1]\) the left supporting segment from \( x \) to \( D \) and 
call \([x, y_2]\) the right supporting segment from \( x \) to \( D \) provided that the order along 
\( \partial \text{conv} \, (y_1, x, y_2) \) in a clockwise direction from \( y_1 \) gives \( y_1, x, y_2 \). A parallel 
definition holds for left and right supporting lines \( L(x, y_1) \) and \( L(x, y_2) \).

We will construct set \( S \) inductively, using an increasing sequence of closed sets. 
Choose a point \( p_1 \) not in \( D \) and in the open convex region bounded by \( T_L \) and \( T_R \).
For the sake of simplicity, let \( p_1 \) be a point on the y axis for \( y > 2 \). Define 
\( S_1 = \text{conv}(p_1 \cup D) \). Let \( w_1 \) be the midpoint of the left supporting segment \([p_1, s_1]\) 
from \( p_1 \) to \( D \). Let \( T_1 \) be the right supporting line from \( w_1 \) to \( D \) at point \( s_2 \). Choose 
the point \( u_1 \) of \( T_1 \) such that \( w_1 \) is between \( u_1 \) and \( s_2 \) and \( \text{dist}(w_1, u_1) \) is half the 
distance from \( w_1 \) to \( T_L \). Then \( (w_1, u_1) \cap S_1 = \emptyset \). Relabel peak \( p_1 \) by \( p_{21} \), peak \( u_1 \) by 
\( p_{22} \equiv b_2 \), and define \( S_2 = \cup \{\text{conv}(p_{2i} \cup D): 1 < i < 2\} \). Label the inc point \( w_1 \) 
of \( S_2 \) by \( s_{21} \), and let \( s_{22} \) be the point of \( D \) such that \([p_{22}, s_{22}] \) is the left supporting 
segment from \( p_{22} \) to \( D \).

Inductively, for \( k > 2 \) assume that consecutive peaks \( p_{k1}, p_{k2}, \ldots, p_{kj} \equiv b_k \) are 
defined, \( j = 2^{k-1} \), that \( S_k = \cup \{\text{conv}(p_{ki} \cup D): 1 < i < j\} \), and that 
\( s_{k1}, s_{k2}, \ldots, s_{kj} \) are defined. We obtain \( S_{k+1} \) as follows. For each \( i, 1 < i < j, \) let 
\( w_{ki} \) be the midpoint of segment \([p_{ki}, s_{ki}]\). Let \( T_{ki} \) be the right supporting line from 
\( w_{ki} \) to \( D \). For \( i \neq j \), choose point \( u_{ki} \) of \( T_{ki} \) whose distance to \( w_{ki} \) is half the distance 
from \( w_{ki} \) to segment \([p_{ki}, p_{k,i+1}]\) along \( T_{ki} \) and \( (p_{ki}, u_{ki}) \cap S_k = \emptyset \). For \( i = j \), 
choose point \( u_{kj} \) of \( T_{kj} \) such that \( \text{dist}(w_{kj}, u_{kj}) = \frac{1}{2} \text{dist}(w_{kj}, T_L) \). (Recall that \( T_L \) is the 
left vertical line of support to \( D \).) Relabel peaks \( p_{k1}, u_{k1}, p_{k2}, u_{k2}, \ldots, p_{kj}, u_{kj} \) by 
\( p_{k+1,1}, p_{k+1,2}, \ldots, p_{k+1,2j} \equiv b_{k+1} \), respectively, and refer to \( p_{k+1,i} \) and \( p_{k+1,i+1} \) as 
consecutive peaks, \( 1 < i < 2j - 1 \). Define \( S_{k+1} = \cup \{\text{conv}(p_{k+1,i} \cup D): 1 < i < 2j\} \). For future reference, notice that for each peak \( p_{k+1,i} \) of \( S_{k+1} \) and for \( M \) a
supporting line from $p_{k+1,i}$ to $D$, sets \{ $p_{k+1,1}, \ldots, p_{k+1,i-1}$ \} and \{ $p_{k+1,i+1}, \ldots, p_{k+1,2j}$ \} lie in opposite open halfplanes determined by $M$. Thus no point $p_{k+1,i}$ is captured by a remaining convex hull \text{conv}(p_{k+1,m} \cup D), m \neq i$. Note also that the slopes of the right supporting lines to $D$ from consecutive peaks of $S_{k+1}$ form an increasing sequence, as do the slopes of the left supporting lines. Finally, observe that \( (p_{k+1,i}, p_{k+1,i+1}) \cap S_{k+1} = \emptyset \) for $1 < i < 2j - 1$.

Continuing our inductive construction, label the inc points of $S_{k+1}$ by \( s_{k+1,1}, s_{k+1,2}, \ldots, s_{k+1,2j-1} \), where $s_{k+1,i}$ is the inc point of \text{conv}(p_{k+1,i} \cup D) \cup \text{conv}(p_{k+1,i+1} \cup D), 1 < i < 2j - 1$. And let $s_{k+1,2j}$ be the point of $D$ such that \[ [p_{k+1,2j}, s_{k+1,2j}] \] is the left supporting segment from $p_{k+1,2j} = b_{k+1}$ to $D$. By our inductive construction, $S_n$ is defined for every natural number $n$. We define set $S_L = \cup_{1 \leq n} S_n$.

Now begin again with set $S_1 = S_1$ and apply the construction above to the right halfplane, using right supporting segments (instead of left), left supporting lines (instead of right), and line $T_R$ (instead of $T_L$). We thereby obtain an increasing sequence of sets \{ $S'_n$ \}, and define set $S_R = \cup_{1 \leq n} S'_n$. Finally, define $S = \text{cl}(S_L \cup S_R)$. Observe that $S_R$ contributes no points to $S_L$ in the left halfplane, $S_L$ contributes none to $S_R$ in the right.

Clearly $S$ is a compact set and $S \neq D$. We assert that $\ker S = D$. It is easy to show that $D \subseteq \ker S$, so we will establish only the reverse inclusion. As a preliminary step, we show that for $p$ one of the peaks defined above, say $p = p_{ki}$, $p$ is an inc point of $S$. Without loss of generality, assume that $p \in S_L$. Let $N$ be any circular neighborhood of $p$. By our construction, $N$ contains another peak, say $q = p_{im}$, where $l > k$ and where $q$ and $\theta$ are in opposite open halfplanes determined by line $L(p_{ki}, s_{ki})$. Moreover, $q$ may be selected so that $p$ and $q$ are consecutive peaks for $S$. Then by a previous comment, $(p, q) \cap S_i = \emptyset$. Select $r$ so that $p, r, q$ are consecutive peaks for $S_i$. For $n > l + 1$, every peak $t \neq p, r, q$ of $S_n$ which lies in the convex region determined by rays $R(\theta, p)$ and $R(\theta, q)$ must lie in the open halfplane determined by $L(p, r)$ and containing $\theta$ and in the open halfplane determined by $L(r, q)$ and containing $\theta$. Thus \text{conv}(t \cup D)$ is disjoint from \text{conv}(p, r, q), and such a peak $t$ cannot contribute any point to \text{conv}(p, r, q)$ in $S_n$ or in $S$. Using previous comments, it is easy to see that the remaining peaks of $S_n$ cannot contribute any point to \text{conv}(p, r, q)$ either, and we conclude that $(p, q) \cap S = \emptyset$. Hence $p$ is indeed an inc point for $S$, and our preliminary result is established.

Now we are able to show that $\ker S \subseteq D$. Let $x \in S \sim D$ to prove that $x \notin \ker S$. Without loss of generality, assume that $x$ is in the closed left halfplane. Recall that, for each $k \geq 2$, $b_k$ denotes the last peak of $S_k$ in our ordering, and clearly the sequence \{ $b_k$ \} converges to a point of $D \cap T_L$. By our construction, it is not hard to see that $T_L \cap S = T_L \cap D$, so $x \notin T_L$. Therefore, there must be some peak $p$ in the open left halfplane such that the right supporting segment from $p$ to $D$ has slope greater than the slope of the right supporting segment $E$ from $x$ to $D$. That is, $x$ and $\theta$ lie in opposite open halfplanes $E_1$ and $E_2$, respectively, determined by the line of $E$. Again by our construction, there is a sequence of
distinct peaks in $E_1$ converging to a point of $E$, and it is not hard to see that this sequence is eventually in $\text{int conv}(p \cup x \cup D)$. Hence for $n$ sufficiently large, $\text{int conv}(p \cup x \cup D)$ necessarily contains peaks of $S_n$, and since such peaks are $\text{linc}$ points of $S$, $\text{int conv}(p \cup x \cup D)$ must contain points in $\sim S$. Therefore, $x \cup D \subset \ker S$, and since $D \subset \ker S$, this implies that $x \not\in \ker S$. We conclude that $\ker S \subset D$ and the sets are equal, completing the proof of Theorem 1.

In case $D$ contains a ray but no line, the argument above may be adapted to obtain the following result.

**Theorem 2.** Let $D$ be any closed unbounded convex set in $\mathbb{R}^2$ which contains no line. Then there is a closed set $S \neq D$ in $\mathbb{R}^2$ with $\ker S = D$.

**Proof.** If $D$ has dimension less than 2, a variation of our opening argument in Theorem 1 yields the result. Hence assume that $D$ is fully 2-dimensional. For convenience, orient $D$ so that $\text{int } D$ contains the negative $y$ axis, $D$ does not contain either the positive or the negative $x$ axis, and $D$ has a horizontal line of support at the origin. For point $x \notin D$, there are two distinct supporting lines from $x$ to $D$ (which may or may not meet $D$). The definitions of left and right supporting lines from $x$ to $D$ (in Theorem 1) may be adapted in an obvious manner to this case. When such a supporting line meets $D$, we may talk about the corresponding left or right supporting segment from $x$ to $D$.

We proceed as follows to define set $S$. In case $D$ is a cone, $D$ may be represented by $\bigcup \{ R(v, c) : c \in C \}$ where $C$ is some compact convex set not containing point $v$. Then it is easy to construct $S$ by creating an $\text{linc}$ point appropriately at $v$. For the remainder of the argument, we will assume that $D$ is not a cone. Select points $z_L$ and $z_R$ in $\text{bdry } D$, $z_L$ in the left open halfplane, $z_R$ in the right. Also, since $D$ is not a cone, we may choose these points so that at least one of the segments $[z_L, \theta]$ and $[z_R, \theta]$ is not in $\text{bdry } D$. Let $T_L$ and $T_R$ denote lines which support $D$ at $z_L$ and $z_R$, respectively, and let $A$ denote the open convex region determined by $T_L$ and $T_R$ and containing $\text{int } D$. Either $A$ will be the interior of a cone whose vertex lies in the upper halfplane or $A$ will be an open parallel strip.

Define $B$ to be the bounded component of $(\text{bdry } D) \sim (T_L \cup T_R)$. By our choice of $z_L$ and $z_R$, it is clear that $B$ is not empty and $\text{cl } B$ contains the origin. Furthermore, if $F$ denotes the component of $A \sim D$ whose boundary contains $B$, then for $x$ in $F$, each supporting segment from $x$ to $D$ necessarily meets $\text{cl } B$. Select point $p_1$ in $F$ and define $S_1 = \text{conv}(p_1 \cup D)$. Using $p_1$, $T_L$, and $T_R$ defined above, employ the inductive construction in Theorem 1. At each stage of the construction, every peak $p_k$ will lie in the region $F$. With minor modifications in terminology, the previous argument may be duplicated to define the closed set $S$ and to verify that $\ker S = D$. This completes the proof of Theorem 2.

If the closed set $D$ contains a line, then no appropriate set $S$ exists, as our final theorem reveals.

**Theorem 3.** Let $D$ be any convex set in $\mathbb{R}^2$ which contains a line $L$. Then $D$ is the kernel of some planar set $S \neq D$ if and only if there is a line $M$ in $\text{bdry } D$ with $M \cap D = \emptyset$. 

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Proof. We dispose of the easy case first. If bdry $D$ contains such a line $M$, select distinct points $p$ and $q$ in $M$. Defining $S = D \cup \{p, q\}$, standard arguments reveal that $D = \ker S$.

To establish the converse statement, assume that no such line $M$ exists. If $D = \mathbb{R}^2$, the result is immediate, so suppose that this is not the case. Then clearly bdry $D = H \cup J$ where $H$ and $J$ are lines parallel to $L$ (and not necessarily distinct). Furthermore, by hypothesis, $H \cap D \neq \emptyset$ and $J \cap D \neq \emptyset$. Let $S$ denote any planar set distinct from $D$ with $D \subseteq \ker S$. (Clearly $\mathbb{R}^2$ itself is such a set.) To finish the proof of Theorem 3, it suffices to show that ker $S \subseteq D$.

In case $S \subseteq \text{cl } D$, then since $H \cap D \neq \emptyset$ and $J \cap D \neq \emptyset$, it is easy to show that $S$ is convex. Thus ker $S = S \subseteq D$, and the theorem is proved. Therefore, throughout the remainder of the argument we may restrict our attention to the case in which $S \nsubseteq \text{cl } D$.

The following notation will be useful. For each point $z$ in $S \sim L$, let $M_z$ denote the line through $z$ and parallel to $L$, with $P_z$ the open parallel strip bounded by $M_z$ and $L$. Notice that since $L \cap \text{cl } D \subseteq \ker S$, $P_z \subseteq S$.

To show that ker $S \subseteq D$, select a point $x$ in $S \sim \text{cl } D \neq \emptyset$. There is a neighborhood $N$ of $x$ disjoint from $\text{cl } D$, and we may choose some point $x'$ in $N \cap P_x \subseteq S \sim \text{cl } D$. We assert that $x' \in \ker S$: Choose $y \in S$ to show that $[x', y] \subseteq S$. If $y \in L \subseteq \ker S$, then it is immediate that $[x', y] \subseteq S$, so we assume that $y \in S \sim L$. Then $P_x \cup P_y \cup L$ is a (not necessarily open) parallel strip in $S$ with $x'$ in its interior and $y$ in its closure. Thus $[x', y] \subseteq P_x \cup P_y \cup L \subseteq S$ and $[x', y] \subseteq S$. We conclude that $x' \in \ker S \sim \text{cl } D \subseteq \ker S \sim D$ and ker $S \subseteq D$. The proof of Theorem 3 is established.

We conclude the planar case with two corollaries. The first may be proved using an argument from Theorem 3 above.

Corollary 1. Let $D$ be any convex set in $\mathbb{R}^2$ which contains a line. Then $D$ is not the kernel of any planar set which property contains $\text{cl } D$.

The second corollary, our characterization theorem, is an immediate consequence of Theorems 1, 2, and 3.

Corollary 2. Let $D$ be a nonempty closed convex subset of $\mathbb{R}^2$. Then $D$ is the kernel of some planar set $S \neq D$ if and only if $D$ contains no line.

Notice that Corollary 2 above fails without the requirement that $D$ be closed, as Theorem 3 reveals.

3. The results in $\mathbb{R}^d$. Theorem 1 may be modified to obtain the following extension from $\mathbb{R}^2$ to $\mathbb{R}^d$.

Theorem 4. Let $D$ be a nonempty compact set in $\mathbb{R}^d$, $d > 2$. Then there is a compact set $S \neq D$ in $\mathbb{R}^d$ with ker $S = D$.

Proof. The inductive construction for $S$ is outlined below. We restrict our attention to $\mathbb{R}^d$ for $d > 3$, and assume that $D$ is fully $d$-dimensional. Consider the family $\mathcal{F}$ of closed halfspaces of the form $[a : a] = \{z \in \mathbb{R}^d \text{ and } a \cdot z > a\}$,
where \( \alpha \) is any \( d \)-dimensional vector having rational coordinates and where \( a \) is any rational scalar. It is known that \( \mathcal{F} \) is an intersectional basis for the set of compact convex subsets of \( \mathbb{R}^d \). Furthermore, if we define \( \mathcal{K} \equiv \{ F \in \mathcal{F} \text{ and } D \subseteq F \} \), then \( D = \bigcap \{ F : F \in \mathcal{K} \} \). Members of \( \mathcal{K} \) may be indexed \( H_1, H_2, \ldots \). For convenience of notation, we will let \( J_i \) denote the supporting hyperplane for \( H_i \), \( 1 < i < \infty \).

To begin the induction, select the first hyperplane \( J_1 \) and choose a point \( x_1 \) in \( J_1 \sim D \neq \emptyset \). Define \( S_1 = \text{conv}(x_1 \cup D) \). Choose a real number \( f_1 \) so that \( 0 < f_1 < \frac{1}{2} \text{dist}(x_1, D) \), and let \( T_1 = \text{conv}(x_1 \cup N(D, f_1)) \), where \( N(D, f_1) = \{ z : \text{dist}(z, D) < f_1 \} \).

Inductively, assume that for \( k > 2 \), \( x_1, \ldots, x_k, S_1, \ldots, S_k, f_1, \ldots, f_k \) and \( T_1, \ldots, T_k \) are defined, where

\[
S_k = \bigcup_{1 < i < k} \{ \text{conv}(x_i \cup D) \},
\]

\[
0 < f_k < \frac{1}{2} \min \{ \text{dist}(x_i, \text{conv}(x_j \cup D)) : i \neq j, 1 < i, j < k \},
\]

\[
T_k = \bigcup_{1 < i < k} \{ \text{conv}(x_i \cup N(D, f_k)) \},
\]

and

\[
x_i \notin \text{conv}(x_j \cup N(D, f_k)) \text{ for } i \neq j, 1 < i, j < k.
\]

We would like for hyperplane \( J_{k+1} \) to meet \( \text{int} \ N(D, f_k) \). If this does not occur, then by our choice of \( \mathcal{K} \), there must be a halfspace \( H_n \) in \( \mathcal{K} \), \( n > k + 1 \), for which the corresponding hyperplane \( J_n \) is parallel to \( J_{k+1} \) and for which \( J_n \cap \text{int} \ N(D, f_k) \neq \emptyset \). Note that \( H_n \subseteq H_{k+1} \). Remove \( J_{k+1} \) and \( H_{k+1} \) from our list and replace them with \( J_n \) and \( H_n \), respectively. Then renumber appropriate \( J \) and \( H \) sets so that their indices are consecutive. (That is, \( J_n \) will be renamed \( J_{k+1} \) and \( J_{n+m} \) will be renamed \( J_{n+m-1} \) for \( m > 1 \). Corresponding \( H \) sets will be renamed, too.)

Now \( J_{k+1} \cap \text{int} \ N(D, f_k) \neq \emptyset \). Since \( d > 3 \), we may prove that \( J_{k+1} \cap T_k \sim S_k \neq \emptyset \), and select point \( x_{k+1} \in J_{k+1} \cap T_k \sim S_k \). Clearly \( x_{k+1} \in \text{conv}(x_i \cup D) \) for \( 1 < i < k \). Show that \( x_i \notin \text{conv}(x_{k+1} \cup D) \) for \( 1 < i < k \), to conclude that \( x_i \notin \text{conv}(x_j \cup D) \) for \( i \neq j, 1 < i, j < k + 1 \).

Define \( S_{k+1} = \bigcup_{1 < i < k+1} \{ \text{conv}(x_i \cup D) \} \). Choose real number \( f_{k+1} \) so that \( 0 < f_{k+1} < \frac{1}{2} \min \{ \text{dist}(x_i, \text{conv}(x_j \cup D)) : i \neq j, 1 < i, j < k + 1 \} \), and let \( T_{k+1} = \bigcup_{1 < i < k+1} \{ \text{conv}(x_i \cup N(D, f_{k+1})) \} \). Notice that by our choice of \( f_{k+1} \), \( x_i \notin \text{conv}(x_j \cup N(D, f_{k+1})) \) for \( i \neq j, 1 < i, j < k + 1 \). Observe that \( T_{k+1} \subseteq T_k \) and that \( \bigcup_{1 < i < k+1} S_i \subseteq \bigcap_{1 < i < k+1} T_i \). By induction, \( S_n \) is defined for every \( 1 < n \), and we let \( S = \text{cl}(\bigcup_{1 < n} S_n) \). Using the facts that \( \{ f_n \} \) converges to zero and \( x_i \notin \text{conv}(x_j \cup N(D, f_n)) \) for \( 1 < i < n, j \neq i \), one may show that each \( x_i \) selected above sees via some \( T_n \) (and hence via \( S \)) only points in the corresponding closed halfspace \( H_i \). Then it is not hard to prove that set \( S \) satisfies the theorem.

We close with an easy analogue of Theorem 3.

**Theorem 5.** Let \( D \) be any convex set in \( \mathbb{R}^d \) which contains a hyperplane. Then \( D \) is the kernel of some set \( S 
eq D \) in \( \mathbb{R}^d \) if and only if there is a line \( L \) in \( \text{bdry} \ D \) with \( L \cap D = \emptyset \).
Corollary 1. Let $D$ be a closed convex subset of $\mathbb{R}^d$ which contains a hyperplane. Then $D$ is not the kernel of any set $S \neq D$ in $\mathbb{R}^d$.

References


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