PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, August 1981

ON A THEOREM OF ARHANGEL'SKIĬ

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ABSTRACT. We define a class of spaces which is more extensive than the class of BCO spaces and which counts among its members some that are not even first countable, and show that this more extensive class of spaces nevertheless intersects the class of paracompact Hausdorff spaces at precisely the class of metrizable spaces as does the class of BCO spaces, thus extending a theorem of Arhangel'skiĭ. We further show that this extension of Arhangel'skiĭ's result has gone the farthest in the sense that any class of spaces that meets the paracompact spaces at precisely the metrizable spaces must, among the Hausdorff spaces, be smaller than the class we have defined.

In 1963, Arhangel'skiĭ [1], [3], [9] proved that paracompact Hausdorff spaces are metrizable if (and only if) they have what he called Bases of Countable Order (BCO's), giving an improvement on Bing's long-standing theorem [2] that says that paracompact Hausdorff spaces are metrizable if (and only if) they are developable, because there are BCO spaces that are not developable, an example of which is ω_1, the space of all countable ordinals. The proof of Bing's theorem is so very straightforward, if one accepts the Metrization Theorem of Bing-Nagata-Smirnov, that Bing wished to and did in the same breath give an improvement on his own theorem as follows: Developable spaces are metrizable if (and only if) they are collectionwise normal. But that is really equal in strength to Arhangel'skiĭ's, because, as Worrell and Wicke [3], [9] later pointed out, collectionwise normality equals paracompactness less θ-refinability, and the property of having a BCO is developability less the same thing. When confronted with this situation, one naturally attempts to really strengthen Bing's theorem in the direction of and beyond Arhangel'skiĭ's, i.e., beyond the strengthened version of Bing, a theorem so central in the development of Hodel's [4] survey of metrization results.

Below, we will define in §1 a class of spaces so much more extensive than the BCO spaces that it encompasses some that are not even first countable, and we will prove in §2 that its intersection with the class of paracompact Hausdorff spaces is the class of metrizable spaces. We will further show in §3 that among the Hausdorff spaces it is the largest such class. In §4, we will give a formally simpler definition of our class of spaces among the Hausdorff spaces.

Received by the editors November 12, 1979 and, in revised form, October 9, 1980; presented to the Society, October 15, 1979.

1980 *Mathematics Subject Classification*. Primary 54D18, 54E35.

Key words and phrases. Paracompactness, BCO, metrizability, largest class to meet the paracompact at the metrizable.

1. Recall that a BCO is a base $\mathfrak{B}$, any strictly decreasing sequence of members of which constitutes a local base at the points of intersection, if there are any such points [**1**].

DEFINITION 1. On a topological space X, we call a base $\mathfrak{B}$ a Z-base (for want of a better name) if any subcollection $\mathcal{G} = \bigcup_{n\in\mathbf{N}} \mathcal{G}_n$ of $\mathfrak{B}$, so constituted that for each $n \in \mathbf{N}$

(i) $\bigcup \mathcal{G}_n = X$,

(ii) there is a function $g_{n+1}\colon \mathcal{G}_{n+1} \to \mathcal{G}_n$ such that

(a) $G \subset g_{n+1}(G)$, for all $G \in \mathcal{G}_{n+1}$, the inclusion being proper unless G properly contains no nonvoid open subsets,

(b) for each $x \in X$, there exists $G_l \in \mathcal{G}_l$, $l \in \mathbf{N}$, such that $x \in G_n = g_{n+1}(G_{n+1})$, and

(c) for each $y \in X$, there is a neighbourhood Ω (of y) such that, for all $G \in \mathcal{G}_{n+1}$,

$$\Omega \subset g_{n+1}(G), \quad \text{if } y \in G,$$
$$\Omega \cap G = \varnothing, \quad \text{if } y \notin g_{n+1}(G),$$

has the property that given $x \in X$ and a neighbourhood U, there exists an $m \in \mathbf{N}$ such that $x \in G \subset g_{m+1}(G) \subset U$, for some $G \in \mathcal{G}_{m+1}$.

A space having a Z-base will be called a Z-space.

Notice that on a T_1-space if (c) of (ii) is missing in the above, we would have defined merely a BCO once again. For, a BCO is certainly such a thing. On the other hand, if $\mathfrak{B}$ is such a thing, then, given a strictly decreasing sequence $\{G_n\}_{n\in\mathbf{N}}$ out of $\mathfrak{B}$ with $x \in X$ in its intersection, $\mathfrak{B}$ being a base, X a T_1-space, there can be constructed a subcollection $\mathcal{G} = \bigcup_{n\in\mathbf{N}} \mathcal{G}_n$ of $\mathfrak{B}$ satisfying (i) and (a) and (b) of (ii) with (b) strengthened to the degree that, for all $n \in \mathbf{N}$, G_n is the *only* member in $\mathcal{G}_n$ to contain x. This $\mathcal{G}$ is a base and therefore $\{G_n\}_{n\in\mathbf{N}}$ is a local base at x.

From the above it is clear that the BCO spaces are all Z-spaces. To show that the class of Z-spaces properly contains the class of BCO spaces, we exhibit the following example of a Z-space that is not even first countable and therefore a fortiori cannot have a BCO.

EXAMPLE. The space ω_2, i.e. the space of all ordinals smaller than ω_2, i.e. of all ordinals the cardinality of which does not exceed that of the first uncountable ordinal ω_1, with the order topology.

About this space we have the following three propositions.

PROPOSITION 1. *The space ω_2 is not first countable and therefore cannot have a BCO.*

PROPOSITION 2. *Let $\mathcal{A} \equiv \{(\alpha, \beta]\colon \alpha < \beta;\ \alpha, \beta \in \omega_2\}$. Covers of ω_2 by members of $\mathcal{A}$ cannot have cushioned refinements* [**7**] *by members of $\mathcal{A}$.*

PROOF. Suppose the contrary. Let $\mathcal{C} \subset \mathcal{A}$ be a cover of ω_2 which has a cushioned refinement $\mathcal{D} \subset \mathcal{A}$, and therefore a cushioning function c from $\mathcal{D}$ into $\mathcal{C}$, such that $\mathrm{Cl} \bigcup \mathcal{D}' \subset \bigcup \mathcal{C}[\mathcal{D}']$ for all $\mathcal{D}' \subset \mathcal{D}$. Let us write $c(\beta)$ for δ, if $(\gamma, \delta] = c((\alpha, \beta])$. Of course $\beta \leqslant c(\beta)$. Let $(\alpha_1, \beta_1]$ be any member of $\mathcal{D}$. Because $\mathcal{D}$ is a

cover, there are a $(\alpha_2, \beta_2] \in \mathfrak{D}$ such that $c(\beta_1) < \beta_2$, and a $(\alpha_3, \beta_3] \in \mathfrak{D}$ such that $c(\beta_2) < \beta_3$, and ... ad infinitum. There is therefore a strictly increasing sequence $c(\beta_1), c(\beta_2), \ldots$ which has a supremum $\beta \in \omega_2$. Now,

$$\beta \notin \cup \mathcal{C}\left[\{(\alpha_n, \beta_n]: n \in \mathbf{N}\}\right],$$

but $\beta \in \mathrm{Cl} \cup \{(\alpha_n, \beta_n]: n \in \mathbf{N}\}$; which contradicts the stated property of c. Hence no cover of ω_2 by members of $\mathcal{Q}$ can have a cushioned refinement by members of $\mathcal{Q}$.

PROPOSITION 3. *The space ω_2 has a Z-base, viz.*, $\mathcal{Q} \equiv \{(\alpha, \beta]: \alpha < \beta;\ \alpha, \beta \in \omega_2\}$.

PROOF. $\mathcal{Q}$ is of course a base (of the topology of ω_2). It can never have any $\mathcal{G}$ satisfying (c) of (ii) in the definition of a Z-base above, which demands that, for all $n \in \mathbf{N}$, $\mathcal{G}_{n+1}$ be a cushioned refinement of $\mathcal{G}_n$, and therefore satisfies the requirements of a Z-base (vacuously).

2. Having delineated the Z-spaces, we are now ready to prove the main theorem.

THEOREM 4. *Paracompact Hausdorff spaces are metrizable if (and only if) they are Z-spaces.*

PROOF. Let X be paracompact. Let $\mathfrak{B}$ be a Z-base on X. Let $\mathcal{G}_1$ be a cover of X by members of $\mathfrak{B}$. Suppose a cover $\mathcal{G}_n$ of X by members of $\mathfrak{B}$ has been chosen. Let $\mathcal{F}_n$ be such a *locally finite* open refinement of $\mathcal{G}_n$ that there is a function h_n: $\mathcal{F}_n \to \mathcal{G}_n$ such that $F \subset \mathrm{Cl}\, F \subset h_n(F)$ for all $F \in \mathcal{F}_n$. Of course, for each $x \in X$ there is a neighbourhood Ω such that for all $F \in \mathcal{F}_n$

$$\Omega \subset h_n(F) \quad \text{if } x \in F,$$
$$\Omega \cap F = \varnothing \quad \text{if } x \notin h_n(F).$$

Let $\mathcal{G}_{n+1}$ be such a refinement of $\mathcal{F}_n$ by members of $\mathfrak{B}$ that there is a function k_{n+1}: $\mathcal{G}_{n+1} \to \mathcal{F}_n$ such that $G \subset k_{n+1}(G)$ for all $G \in \mathcal{G}_{n+1}$, the inclusion being proper unless G contains no proper open subsets. The subcollection $\mathcal{G} = \cup_{n \in \mathbf{N}} \mathcal{G}_n$ of $\mathfrak{B}$ so obtained clearly satisfies (i) and (a) and (c) of (ii) in the definition of a Z-base, if we let $g_{n+1} = h_n \circ k_{n+1}$ for all $n \in \mathbf{N}$. That it also satisfies (b) of (ii) can be seen from the following. For all $n \in \mathbf{N}$, $\mathcal{F}_n$ is locally finite and therefore point-finite, and there is $f_{n+1} \equiv (k_{n+1} \circ h_{n+1})$: $\mathcal{F}_{n+1} \to \mathcal{F}_n$ so that $F \subset f_{n+1}(F)$ for all $F \in \mathcal{F}_{n+1}$. A lemma of D. König [6] (see also Theorem 114 of Chapter I of Moore [8]) applies. Therefore given $x \in X$ there is $F_n \in \mathcal{F}_n$ for all $n \in \mathbf{N}$ such that $x \in F_{n+1} \subset f_{n+1}(F_{n+1}) = F_n$, for all $n \in \mathbf{N}$. If for every $n \in \mathbf{N}$ we let $G_n = h_n(F_n)$, we have $x \in G_n \in \mathcal{G}_n$ and

$$\begin{aligned} g_{n+1}(G_{n+1}) &= g_{n+1}(h_{n+1}(F_{n+1})) = (h_n \circ k_{n+1} \circ h_{n+1})(F_{n+1}) \\ &= h_n(f_{n+1}(F_{n+1})) = h_n(F_n) = G_n, \end{aligned}$$

for every $n \in \mathbf{N}$; and (b) of (ii) is also satisfied.

Because $\mathfrak{B}$ is a Z-base, such a $\mathcal{G} = \cup_{n \in \mathbf{N}} \mathcal{G}_n$ has the additional property that for each $x \in X$ and each neighbourhood U there is $m \in \mathbf{N}$ such that

$$x \in G \subset g_{m+1}(G) \subset U, \quad \text{for some } G \in \mathcal{G}_{m+1}.$$

According to Theorem 2.1 of [5], X is metrizable. Q.E.D.

Note that in the proof of Theorem 4 above, the construction of $\mathcal{G}$ depends only on the paracompactness of the space and the fact that $\mathcal{B}$ is a base. We therefore have the following, also subsuming the result of Arhangel'skiĭ.

THEOREM 5. *A paracompact Hausdorff space X is metrizable if (and only if) on X there is such a base $\mathcal{B}$ that every strictly decreasing sequence $\{B_n\}_{n\in\mathbf{N}}$ of its members constitutes a local base at the points of intersection when in addition it is also true that, for all $n \in \mathbf{N}$, $B_n \supset \mathrm{Cl}\, B_{n+1}$.*

REMARKS. From the conspicuous presence everywhere above of the nesting condition, one may have the impression that it is essential for our conclusions, while evidently we could completely drop (b) of (ii) in the definition of a Z and have (even more readily) Theorem 4 with Z-bases thus defined.

3. We will now show that among the Hausdorff spaces the class of Z-spaces is the most extensive class whose intersection with the class of paracompact spaces is the class of metrizable spaces.

THEOREM 6. *All Hausdorff spaces are either* (i) *metrizable, or* (ii) *paracompact but not Z-spaces, or* (iii) *Z-spaces but not paracompact. The three possibilities are mutually exclusive.*

PROOF. In view of Theorem 4 above, it suffices to prove that if a space X does not have a Z-base, it is paracompact. Note that if X does not have a Z-base, then out of members of *every* base $\mathcal{B}$ of X, we should be able to erect a $\mathcal{G} = \bigcup_{n\in\mathbf{N}} \mathcal{G}_n$ with the properties (i) and (a), (b) and (c) of (ii) set out in the definition of a Z-base.

Let $\mathcal{C}$ be any open cover of X. It defines a base $\mathcal{A} \equiv \{A\colon A \subset C$ for some $C \in \mathcal{C}$, A is open$\}$, from which, if X does not have a Z-base, we can erect a $\mathcal{G}$ of the description above and bring in a cushioned refinement of $\mathcal{C}$ in the form of $\mathcal{G}_2$ (which is a cushioned refinement of $\mathcal{G}_1$, itself refining $\mathcal{C}$). X is therefore paracompact if it does not have a Z-base. Q.E.D.

4. One complaint about Z-bases, if there is any, might be the lengthy statement involved in its definition. We will look at another class of spaces with a definition that is formally simpler to state and show in the sequel its equivalence, among Hausdorff spaces, to the class of Z-spaces.

DEFINITION 2. Given open covers $\mathcal{C}$ and $\mathcal{D}$ of a space X, we say $\mathcal{D}$ is *strictly cushioned* in $\mathcal{C}$ if there is a *strictly cushioning* function $\mathcal{SC}\colon \mathcal{D} \to \mathcal{C}$ such that

(i) $\mathrm{Cl} \bigcup \mathcal{D}' \subset \bigcup \mathcal{SC}[\mathcal{D}']$, for all $\mathcal{D}' \subset \mathcal{D}$, and

(ii) $D \subset \mathcal{SC}(D)$, for all $D \in \mathcal{D}$, the inclusion being proper unless D properly contains no nonvoid open subsets.

DEFINITION 3. On a space X, we call a base $\mathcal{B}$ a Y-base (for want of a better name) if any subcollection $\mathcal{G} = \bigcup_{n\in\mathbf{N}} \mathcal{G}_n$ of $\mathcal{B}$, so constituted that, for each $n \in \mathbf{N}$, $\mathcal{G}_{n+1}$ is strictly cushioned in $\mathcal{G}_n$, with a strictly cushioning function $\mathcal{SC}_{n+1}$,

has the property that given $x \in X$ and a neighbourhood U, there exists an $m \in \mathbf{N}$ such that

$$x \in G \subset \mathcal{SC}_{m+1}(G) \subset U, \quad \text{for some } G \in \mathcal{G}_{m+1}.$$

A space having a Y-base will be called a Y-space.

Clearly (on a space X) a Y-base is a Z-base. Besides, an examination of the proof of Theorem 4 will show that the class of Y-spaces intersects the paracompact Hausdorff at precisely the metrizable spaces. Furthermore, a look at the proof of Theorem 6 will show that a space is also paracompact if it does not have a Y-base. Therefore among Hausdorff spaces, the class of Z-spaces coincides with the class of Y-spaces.

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