AN ABSTRACT FIXED POINT
THEOREM FOR NONEXPANSIVE MAPPINGS

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Abstract. A class $S$ of subsets of a bounded metric space is said to be normal if each member of $S$ contains a nondiametral point. An induction proof is given for the following. Suppose $M$ is a nonempty bounded metric space which contains a class $S$ of subsets which is countably compact, normal, stable under arbitrary intersections, and which contains the closed balls in $M$. Then every nonexpansive self-mapping of $M$ has a fixed point.

In [7], J. P. Penot presents an abstract version of the writer's fixed point theorem of [3] for nonexpansive mappings. Penot's result is based essentially upon the original line of argument (which uses Zorn's lemma). We show here that a different approach yields the abstract result under even weaker assumptions.

Let $(M, d)$ be a metric space; for a subset $D$ of $M$, let

\[
\delta(D) = \sup \{d(u, v) : u, v \in D\},
\]

\[
r_u(D) = \sup \{d(u, v) : v \in D\} \quad (u \in D),
\]

\[
r(D) = \inf \{r_u(D) : u \in D\},
\]

and

\[
h(D) = \begin{cases} 
\frac{r(D)}{\delta(D)} & \text{if } \delta(D) > 0, \\
1 & \text{if } \delta(D) = 0.
\end{cases}
\]

Definition [7]. A class $S$ of subsets of $M$ is said to be normal if for each $D \in S$, $\delta(D) > 0 \Rightarrow h(D) \in (0, 1)$. The class $S$ is said to be [countably] compact if each [countable] subfamily of $S$ which has the finite intersection property has nonvoid intersection.

Recall that a mapping $T : M \to M$ is said to be nonexpansive if $d(T(u), T(v)) < d(u, v)$, $u, v \in M$. We use $B(u; r)$ to denote the closed ball centered at $u \in M$ with radius $r > 0$.

Theorem 1. Let $(M, d)$ be a nonempty bounded metric space and suppose $M$ contains a class $S$ of subsets which is countably compact, stable under arbitrary intersections, and normal. Suppose further that $S$ contains the closed balls of $M$. Then every nonexpansive mapping $T$ of $M$ into itself has a fixed point.
The above differs from Penot's result in that countable compactness is assumed rather than compactness. We base our proof upon the following abstraction of a lemma due to Gillespie and Williams [2].

**Lemma.** Let \((M, d)\) be a nonempty bounded metric space and let \(\mathcal{S}\) be a class of subsets of \(M\) which contains the closed balls of \(M\) and which is stable under arbitrary intersections. Suppose \(T: M \to M\) is nonexpansive. Then for each \(\epsilon > 0\) there exists a nonempty set \(M(\epsilon) \in \mathcal{S}\) such that \(T: M(\epsilon) \to M(\epsilon)\) and for which \(\delta(M(\epsilon)) < (h(M) + \epsilon)\delta(M)\).

**Proof.** If \(\delta(M) = 0\), take \(M(\epsilon) = M\). Otherwise, construct \(M(\epsilon)\) as follows. Let \(\rho = (h(M) + \epsilon)\delta(M)\). By the definition of \(h\), the set \(\mathcal{C} = \{z \in M: M \subset B(z; \rho)\}\) is nonempty. Let

\[\mathcal{F} = \{D \in \mathcal{S}: \mathcal{C} \subset D, T: D \to D\}\]

and let \(L = \cap \mathcal{F}\). Note that \(\mathcal{F} \neq \emptyset\) since \(M \in \mathcal{F}\). Also \(L \in \mathcal{S}\), \(\mathcal{C} \subset L\), and \(T: L \to L\). Let \(A = \mathcal{C} \cup T(L)\). Then \(A \subset L\); thus \(\text{cov}(A) = \cap \{D: D \in \mathcal{S}, A \subset D\}\) \(\subset L\) from which \(T(\text{cov}(A)) \subset T(L) \subset A \subset \text{cov}(A)\), proving \(\text{cov}(A) \in \mathcal{F}\). Therefore \(\text{cov}(A) = L\).

Now let

\[M(\epsilon) = \{x \in L: L \subset B(x; \rho)\}\]

Then \(\mathcal{C} \subset M(\epsilon)\), so \(M(\epsilon) \neq \emptyset\). Also if \(x \in M(\epsilon)\), then \(T(x) \in L\) and for each \(y \in L\), \(d(T(x), T(y)) < d(x, y) < \rho\). Furthermore if \(z \in \mathcal{C}\), \(d(T(x), z) < \rho\) (because \(M \subset B(z; \rho)\)). This proves that \(A \subset B(T(x); \rho)\) which in turn implies \(L = \text{cov}(A) \subset B(T(x); \rho)\), i.e., \(T: M(\epsilon) \to M(\epsilon)\). Finally,

\[M(\epsilon) = \left( \bigcap_{u \in L} B(u; \rho) \right) \cap L.

Thus \(M(\epsilon)\) is the intersection of sets in \(\mathcal{S}\); hence \(M(\epsilon) \in \mathcal{S}\). Since obviously \(\delta(M(\epsilon)) < \rho\), this completes the proof.

**Proof of Theorem 1.** Let \(\mathcal{M} = \{D \in \mathcal{S}: D \neq \emptyset, T: D \to D\}\) and for each \(D \in \mathcal{M}\), let \(\delta_0(D) = \inf\{\delta(F): F \in \mathcal{M}, F \subset D\}\). Set \(D_1 = M\), and with \(D_1, \ldots, D_n\) given, select \(D_{n+1} \in \mathcal{M}\) so that \(D_{n+1} \subset D_n\) and

\[\delta(D_{n+1}) < \delta_0(D_n) + 1/n\]

Let \(C = \bigcap_{n=1}^{\infty} D_n\). Then \(C \in \mathcal{S}\) and by countable compactness, \(C \neq \emptyset\). Thus \(C \in \mathcal{M}\). By the lemma we now have for each \(\epsilon > 0\) and \(n \in \mathbb{N}\),

\[\delta(C) - 1/n < \delta(D_{n+1}) - 1/n < \delta_0(D_n) < \delta(C) < (h(C) + \epsilon)\delta(C)\]

Letting \(n \to \infty\),

\[\delta(C) < (h(C) + \epsilon)\delta(C)\]

Since this is true for each \(\epsilon > 0\),

\[\delta(C) < h(C)\delta(C),\]

and because \(\mathcal{S}\) is normal this in turn implies \(\delta(C) = 0\). Therefore \(C = \{x\}\) with \(T(x) = x\).
Remark 1. A constructive proof of Theorem 1 can be given by using the lemma in conjunction with the above, but with $\varepsilon$ depending on $D$.

Remark 2. In [1], Fuchssteiner gives another constructive proof of the theorem of [3]. (Also see Lim [5].) Our method seems more direct. Fuchssteiner's approach invokes a fixed point theorem of Zermelo, and consequently an adaptation of that approach to the present setting would require the assumption of compactness on $\mathcal{S}$ rather than countable compactness.

Remark 3. Suppose $X$ is a Banach space with $\tau$ any topology on $X$ for which the norm closed balls are $\tau$-closed, and say that a $\tau$-closed convex subset $K$ of $X$ has $\tau$-normal structure if for each bounded $\tau$-closed convex subset $D$ of $K$, $\delta(D) > 0 \Rightarrow h(D) \in (0, 1)$. Then the following is an immediate special case of Theorem 1.

Theorem 2. Let $K$ be a nonempty bounded $\tau$-closed convex subset of $X$ which has $\tau$-normal structure and which is countably compact in the $\tau$-topology. Suppose $T: K \to K$ is nonexpansive. Then $T$ has a fixed point in $K$.

Proof. Take $\mathcal{S}$ to be the family of $\tau$-closed convex subsets of $K$ and apply Theorem 1.

If $\tau$ is the weak topology on $X$, the above reduces to the original theorem of [3]. Theorem 2 is also known for $X$ a conjugate space and $\tau$ the weak* topology on $X$ ([4], cf. [6]). However, because of the Eberlein-Smulian Theorem and the Alaoglu Theorem, in neither of these instances does the assumption of countable compactness yield greater generality.

References

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