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AN ABSTRACT FIXED POINT THEOREM FOR NONEXPANSIVE MAPPINGS

W. A. KIRK[1]

ABSTRACT. A class $\mathcal{S}$ of subsets of a bounded metric space is said to be normal if each member of $\mathcal{S}$ contains a nondiametral point. An induction proof is given for the following. Suppose M is a nonempty bounded metric space which contains a class $\mathcal{S}$ of subsets which is countably compact, normal, stable under arbitrary intersections, and which contains the closed balls in M. Then every nonexpansive self-mapping of M has a fixed point.

In [7], J. P. Penot presents an abstract version of the writer's fixed point theorem of [3] for nonexpansive mappings. Penot's result is based essentially upon the original line of argument (which uses Zorn's lemma). We show here that a different approach yields the abstract result under even weaker assumptions.

Let (M, d) be a metric space; for a subset D of M, let

$$\begin{aligned} \delta(D) &= \sup\{d(u, v): u, v \in D\}, \\ r_u(D) &= \sup\{d(u, v): v \in D\} \qquad (u \in D), \\ r(D) &= \inf\{r_u(D): u \in D\}, \end{aligned}$$

and

$$h(D) = \begin{cases} r(D)/\delta(D) & \text{if } \delta(D) > 0, \\ 1 & \text{if } \delta(D) = 0. \end{cases}$$

DEFINITION [7]. A class $\mathcal{S}$ of subsets of M is said to be *normal* if for each $D \in \mathcal{S}$, $\delta(D) > 0 \Rightarrow h(D) \in (0, 1)$. The class $\mathcal{S}$ is said to be [*countably*] *compact* if each [countable] subfamily of $\mathcal{S}$ which has the finite intersection property has nonvoid intersection.

Recall that a mapping $T: M \to M$ is said to be *nonexpansive* if $d(T(u), T(v)) < d(u, v)$, $u, v \in M$. We use $B(u; r)$ to denote the closed ball centered at $u \in M$ with radius $r > 0$.

THEOREM 1. *Let (M, d) be a nonempty bounded metric space and suppose M contains a class $\mathcal{S}$ of subsets which is countably compact, stable under arbitrary intersections, and normal. Suppose further that $\mathcal{S}$ contains the closed balls of M. Then every nonexpansive mapping T of M into itself has a fixed point.*

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The above differs from Penot's result in that countable compactness is assumed rather than compactness. We base our proof upon the following abstraction of a lemma due to Gillespie and Williams [2].

LEMMA. *Let (M, d) be a nonempty bounded metric space and let $\mathbb{S}$ be a class of subsets of M which contains the closed balls of M and which is stable under arbitrary intersections. Suppose $T: M \to M$ is nonexpansive. Then for each $\varepsilon > 0$ there exists a nonempty set $M(\varepsilon) \in \mathbb{S}$ such that $T: M(\varepsilon) \to M(\varepsilon)$ and for which $\delta(M(\Sigma)) \leqslant (h(M) + \varepsilon)\delta(M)$.*

PROOF. If $\delta(M) = 0$, take $M(\varepsilon) = M$. Otherwise, construct $M(\varepsilon)$ as follows. Let $\rho = (h(M) + \varepsilon)\delta(M)$. By the definition of h, the set $\mathcal{C} = \{z \in M: M \subset B(z; \rho)\}$ is nonempty. Let

$$\mathcal{F} = \{D \in \mathbb{S} : \mathcal{C} \subset D, T: D \to D\}$$

and let $L = \bigcap \mathcal{F}$. Note that $\mathcal{F} \neq \emptyset$ since $M \in \mathcal{F}$. Also $L \in \mathbb{S}$, $\mathcal{C} \subset L$, and $T: L \to L$. Let $A = \mathcal{C} \cup T(L)$. Then $A \subset L$; thus $\operatorname{cov}(A) = \bigcap \{D: D \in \mathbb{S}, A \subset D\} \subset L$ from which $T(\operatorname{cov}(A)) \subset T(L) \subset A \subset \operatorname{cov}(A)$, proving $\operatorname{cov}(A) \in \mathcal{F}$. Therefore $\operatorname{cov}(A) = L$.

Now let

$$M(\varepsilon) = \{x \in L: L \subset B(x; \rho)\}.$$

Then $\mathcal{C} \subset M(\varepsilon)$, so $M(\varepsilon) \neq \emptyset$. Also if $x \in M(\varepsilon)$, then $T(x) \in L$ and for each $y \in L$, $d(T(x), T(y)) \leqslant d(x, y) \leqslant \rho$. Furthermore if $z \in \mathcal{C}$, $d(T(x), z) \leqslant \rho$ (because $M \subset B(z; \rho)$). This proves that $A \subset B(T(x); \rho)$ which in turn implies $L = \operatorname{cov}(A) \subset B(T(x); \rho)$, i.e., $T: M(\varepsilon) \to M(\varepsilon)$. Finally,

$$M(\varepsilon) = \left\{ \bigcap_{u \in L} B(u; \rho) \right\} \cap L.$$

Thus $M(\varepsilon)$ is the intersection of sets in $\mathbb{S}$; hence $M(\varepsilon) \in \mathbb{S}$. Since obviously $\delta(M(\varepsilon)) \leqslant \rho$, this completes the proof.

PROOF OF THEOREM 1. Let $\mathfrak{M} = \{D \in \mathbb{S}: D \neq \emptyset, T: D \to D\}$ and for each $D \in \mathfrak{M}$, let $\delta_0(D) = \inf\{\delta(F): F \in \mathfrak{M}, F \subset D\}$. Set $D_1 = M$, and with $D_1, \ldots, D_n$ given, select $D_{n+1} \in \mathfrak{M}$ so that $D_{n+1} \subset D_n$ and

$$\delta(D_{n+1}) \leqslant \delta_0(D_n) + 1/n.$$

Let $C = \bigcap_{n=1}^{\infty} D_n$. Then $C \in \mathbb{S}$ and by countable compactness, $C \neq \emptyset$. Thus $C \in \mathfrak{M}$. By the lemma we now have for each $\varepsilon > 0$ and $n \in \mathbf{N}$,

$$\delta(C) - 1/n \leqslant \delta(D_{n+1}) - 1/n \leqslant \delta_0(D_n) \leqslant \delta(C(\varepsilon)) \leqslant (h(C) + \varepsilon)\delta(C).$$

Letting $n \to \infty$,

$$\delta(C) \leqslant (h(C) + \varepsilon)\delta(C).$$

Since this is true for each $\varepsilon > 0$,

$$\delta(C) \leqslant h(C)\delta(C),$$

and because $\mathbb{S}$ is normal this in turn implies $\delta(C) = 0$. Therefore $C = \{x\}$ with $T(x) = x$.

REMARK 1. A constructive proof of Theorem 1 can be given by using the lemma in conjuction with the above, but with ε depending on D.

REMARK 2. In [1], Fuchssteiner gives another constructive proof of the theorem of [3]. (Also see Lim [5].) Our method seems more direct. Fuchssteiner's aproach invokes a fixed point theorem of Zermelo, and consequently an adaptation of that approach to the present setting would require the assumption of compactness on $\mathbb{S}$ rather than countable compactness.

REMARK 3. Suppose X is a Banach space with τ any topology on X for which the norm closed balls are τ-closed, and say that a τ-closed convex subset K of X has *τ-normal structure* if for each bounded τ-closed convex subset D of K, $\delta(D) > 0 \Rightarrow h(D) \in (0, 1)$. Then the following is an immediate special case of Theorem 1.

THEOREM 2. *Let K be a nonempty bounded τ-closed convex subset of X which has τ-normal structure and which is countably compact in the τ-topology. Suppose T: $K \to K$ is nonexpansive. Then T has a fixed point in K.*

PROOF. Take $\mathbb{S}$ to be the family of τ-closed convex subsets of K and apply Theorem 1.

If τ is the weak topology on X, the above reduces to the original theorem of [3]. Theorem 2 is also known for X a conjugate space and τ the weak* topology on X ([4], cf. [6]). However, because of the Eberlein-Smulian Theorem and the Alaoglu Theorem, in neither of these instances does the assumption of countable compactness yield greater generality.

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DEPARTMENT OF MATHEMATICS, UNVIERSITY OF IOWA, IOWA CITY, IOWA 52242.