PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, August 1981

SHORTER NOTES

The purpose of this department is to publish very short papers of unusually elegant and polished character, for which there is no other outlet.

TOTALLY REAL KLEIN BOTTLES IN C^2

WALTER RUDIN[1]

ABSTRACT. It is proved that C^2 contains totally real compact submanifolds that are nonorientable, being homeomorphic to a Klein bottle.

A smooth manifold M in C^n is said to be *totally real* if none of its tangent spaces $T_p(M)$ (where $p \in M$) contains a complex subspace. In other words, $T_p(M)$ and $iT_p(M)$ are to have only 0 in common. When the (real) dimension of M is n (briefly, when M is an n-manifold) this amounts to requiring that the C-span of every $T_p(M)$ is all of C^n.

The totally real submanifolds of C^n play an important role in problems about approximation of continuous functions by holomorphic ones. See **[3]**, where further references are given, and also **[1]**, where these manifolds occur in a different context.

The most obvious totally real n-manifold in C^n is R^n, the set of all points in C^n whose coordinates are real. As far as compact manifolds are concerned, Wells **[2]** has proved that the Euler characteristic of every compact *orientable* totally real n-manifold in C^n is zero. Among the compact orientable 2-manifolds there is therefore only one, namely the torus, that has totally real embeddings in C^2. The set of all points $(e^{i\theta}, e^{i\varphi})$, where $(\theta, \varphi) \in R^2$, is the standard example of such an embedding of the torus T^2.

However, C^2 also contains compact totally real 2-manifolds that are *not* orientable, a fact that has apparently escaped earlier notice.

THEOREM. *There is an entire map* F: $C^2 \to C^2$ *such that* $F(R^2)$ *is totally real and homeomorphic to a Klein bottle.*

PROOF. Pick constants $a, b,\ a > b > 0$, put

$$g(\theta, \varphi) = (a + b \cos \varphi)e^{i\theta}, \qquad h(\varphi) = \sin \varphi + i \sin 2\varphi,$$

Received by the editors November 21, 1980.

1980 *Mathematics Subject Classification.* Primary 32E30; Secondary 57R40.

[1]This research was partially supported by NSF Grant MCS 78-06860 and by the William F. Vilas Trust Estate.

and define $F(\theta, \varphi) = (z, w)$ by

$$z = g^2(\theta, \varphi), \qquad w = g(\theta, \varphi)h(\varphi).$$

Then $F(\theta, \varphi + 2\pi) = F(\theta, \varphi) = F(\theta + \pi, -\varphi)$, for all $(\theta, \varphi) \in \mathbf{C}^2$. To prove that F separates all other pairs of points in the region

$$Q = \{(\theta, \varphi): -\pi \leqslant \theta < \pi, -\pi \leqslant \varphi < \pi\} \subset R^2,$$

pick $(z, w) \in F(Q)$ and note that

$$a + b \cos \varphi = |z|^{1/2},$$

so that z determines $\cos \varphi$ and $e^{2i\theta}$. This gives two choices for $e^{i\theta}$. Once one of these is made, $\sin \varphi$ is determined by w.

We conclude that F is 2-to-1 on Q, and that $K = F(R^2)$ is a Klein bottle. [Note the minus sign in $F(\theta, \varphi) = F(\theta + \pi, -\varphi)$.]

For each $(\theta, \varphi) \in R^2$, the column vectors

$$\begin{pmatrix} z_\theta \\ w_\theta \end{pmatrix} = \begin{pmatrix} 2gg_\theta \\ hg_\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_\varphi \\ w_\varphi \end{pmatrix} = \begin{pmatrix} 2gg_\varphi \\ hg_\varphi + gh' \end{pmatrix}$$

are tangent to K at $F(\theta, \varphi)$. They are linearly independent (over $\mathbf{C}$) since

$$z_\theta w_\varphi - z_\varphi w_\theta = 2g^2 g_\theta h' = 2ig^3 h' \neq 0.$$

[Note that $h'(\varphi) \neq 0$ for all real φ.] The $\mathbf{C}$-span of these tangent vectors is therefore all of $\mathbf{C}^2$, so that K is totally real.

References

1. S. I. Pinčuk, *A boundary uniqueness theorem for holomorphic functions of several complex variables*, Mat. Z. **15** (1974), 205–212 = Math. Notes **15** (1974), 116–120.

2. R. O. Wells, *Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles*, Math. Ann. **179** (1969), 123–129.

3. ______, *Real analytic subvarieties and holomorphic approximation*, Math. Ann. **179** (1969), 130–141.

Department of Mathematics, University of Wisconsin-Madison, Madison, Wisconsin 53706