THE FRÉCHET SPACE $\omega$

ADmits a strictly stronger separable
and quasicomplete locally convex topology

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Let $\mathcal{L}$ denote the class of all locally convex Hausdorff spaces $(E, \mathcal{I})$ with the following property: Every locally convex Hausdorff topology $\mathcal{S} \subset \mathcal{I}$ on $E$ has the same subfamily summable sequences as $\mathcal{I}$. Several articles have been devoted to the investigation of the richness of $\mathcal{L}$, e.g., Kalton [4], Labuda [6], [7], Graves [3]; see also the references in [3]. For example, $\mathcal{L}$ contains every fully complete locally convex space which does not contain $1^\infty$ [6, p. 219, (8)], hence every separable Fréchet space. E. Thomas asked in a letter of 1976 whether $\mathcal{L}$ even contains every separable quasicomplete space. This note provides a negative answer to this question.

We will use the following results about separability which we prove for general topological vector spaces.

**Lemma.** Every finite codimensional linear subspace $H$ in a separable topological vector space $E$ is separable.

**Proof.** We may at once assume that $H = \ker f$, where $f$ is a discontinuous linear form on $E$.

$E$ contains a dense linear subspace $L$ of countable dimension. For every $x \in E$ let $L_x$ denote the linear span of $L \cup \{x\}$. We denote the topology of $E$ by $\mathcal{I}$. The strongest linear topology $\mathcal{S}$ on $E$ such that for every $x \in E$, the relative topologies $\mathcal{S}|L_x$ and $\mathcal{I}|L_x$ coincide, is clearly stronger than $\mathcal{I}$. Moreover $\mathcal{S}|L = \mathcal{I}|L$ and $L$ is dense in $(E, \mathcal{S})$, hence $\mathcal{I} = \mathcal{S}$ by [2, p. 349, Lemma 1]. Since $f$ is discontinuous we deduce that for some $z \in E$ the restriction $f|L_z$ is discontinuous, whence $H \cap L_z$ is dense in $L_z$. Thus $H \cap L_z$ is dense in $E$ and hence dense in $H$. Since $H \cap L_z$ is of countable dimension, we have proved that $H$ is separable. □

(For a locally convex space $E$, a somewhat technical proof of the lemma has been given by Valdivia in [8, p. 195, Lemma 2].)

**Proposition.** Let $(E, \mathcal{I})$ be a separable topological vector space over $K \in \{\mathbb{R}, \mathbb{C}\}$ and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of linear forms on $E$. Then the initial topology $\mathcal{T}$ on $E$ with respect to the identity map $\text{id}: E \to (E, \mathcal{I})$ and all the functionals $f_n: E \to K$ $(n \in \mathbb{N})$ is again separable.
Proof. For every \( n \in \mathbb{N} \), the space \( E \) provided with the initial topology \( \mathfrak{F}_n \) with respect to id: \( E \to (E, \mathcal{I}) \) and \( f_i: E \to K \) \( (1 \leq i \leq n) \), is the topological direct sum of \( (\bigcap_{1 \leq i \leq n} \ker f_i, \mathcal{I}| \bigcap_{1 \leq i \leq n} \ker f_i) \) and a finite dimensional linear subspace, hence separable according to the lemma. Since \( \mathfrak{F}_n \subset \mathfrak{F}_{n+1} \) \( (n \in \mathbb{N}) \) and \( \mathfrak{F} \) equals the supremum \( \bigvee_{n \in \mathbb{N}} \mathfrak{F}_n \), we obtain the separability of \( (E, \mathfrak{F}) \). \( \square \)

The separable Fréchet space \( \omega := K^\mathbb{N} \) provided with the product topology \( \mathfrak{B} \), clearly carries the initial topology with respect to the sequence of linear forms \( p_n: \omega \to K \), \( (x_m)_{m \in \mathbb{N}} \mapsto x_n \), \( (n \in \mathbb{N}) \). Thus we get the following:

Corollary. For every separable linear topology \( \mathcal{I} \) on \( \omega \) the supremum \( \mathcal{I} \vee \mathfrak{B} \) is again separable.

Remark. We mention that the supremum of two separable linear topologies need not be separable. In fact, let \( (E, \mathcal{I}) \) be a separable locally convex space containing a nonseparable linear subspace \( L \). Choose a linear subspace \( M \subset E \) such that \( L \cap M = \{0\} \) and \( L + M = E \). Then the initial topology \( \mathfrak{S} \) on \( E \) with respect to \( j: E \to (E, \mathcal{I}) \), \( j(x + y) := x - y \ (x \in L, y \in M) \) is also separable. One verifies without difficulty that \( (E, \mathcal{I} \vee \mathfrak{S}) \) is the topologically direct sum of \( (L, \mathcal{I}|L) \) and \( (M, \mathcal{I}|M) \), hence not separable.

Example. We consider the noncomplete separable Montel space \( X \) constructed by Amemyia, Kömura [1] (cf. also Knowles, Cook [5]), whose dimension is not less than the dimension of \( \omega \) and in which every bounded subset has a finite dimensional linear span (see [1], [5]). Consequently there exists an injective linear map \( f: \omega \to X \) with separable range. Let \( \mathcal{I} \) denote the initial topology on \( \omega \) with respect to \( f: \omega \to X \), which is clearly locally convex.

On account of the corollary, \( (\omega, \mathcal{I} \vee \mathfrak{B}) \) is separable. Moreover, every bounded set in \( (\omega, \mathcal{I} \vee \mathfrak{B}) \) has finite dimensional linear span, whence in particular, \( (\omega, \mathcal{I} \vee \mathfrak{B}) \) is quasicomplete.

Finally, the sequence \( (e_n)_{n \in \mathbb{N}} \) of unit vectors \( e_n = (\delta_{mn})_{m \in \mathbb{N}} \in \omega \) is subfamily summable in \( (\omega, \mathfrak{B}) \), but not bounded, hence not summable, in \( (\omega, \mathcal{I} \vee \mathfrak{B}) \). Thus \( (\omega, \mathcal{I} \vee \mathfrak{B}) \not\in \mathfrak{E} \).

References


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