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A COUNTEREXAMPLE TO THE UNIMODULAR CONJECTURE ON FINITELY GENERATED DIMENSION GROUPS

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ABSTRACT. We give a series of examples of simple finitely generated dimension groups which cannot be obtained as the inductive limit of a system

$$\mathbf{Z}^r \xrightarrow{A_1} \mathbf{Z}^r \xrightarrow{A_2} \cdots \mathbf{Z}^r \xrightarrow{A_n} \cdots,$$

where each A_n is a unimodular matrix whose entries are nonnegative integers.

1. In this note we are concerned with ordered groups G of the following form. G is equal to $\mathbf{Z}^r$ as an abelian group only, for some $r \in \mathbf{N}$, and there exists a set $\{a^{(1)}, \ldots, a^{(p)}\}$ of linearly independent vectors in $(\mathbf{R}^r)^+$ such that the positive cone G^+ of G is given by

$$G^+ = \{z \in \mathbf{Z}^r / \langle a^{(i)}, z\rangle > 0;\ i = 1, \ldots, p\} \cup \{0\}.$$

G is a dimension group if it satisfies the Riesz interpolation property (for the definitions concerning dimension groups we refer to [1], [2]). Effros and Shen conjectured in [3] (see also [1]) that if G is a dimension group, then there exists a sequence $A_1, A_2, \ldots$ in the set $\mathrm{GL}(r, \mathbf{Z})^+$ of all unimodular matrices whose entries are nonnegative integers, such that

$$G^+ = \bigcup_{n=1}^{\infty} (A_n \ldots A_1)^{-1}(\mathbf{Z}^r)^+ .$$

In [4] we have shown that the conjecture is true if G is simple (i.e. G^+ has no nontrivial faces) and $p = 1$ holds. Using the theory of diophantine approximation we will show in the sequel that for $p = r - 1$ ($r \geqslant 3$) there exist simple dimension groups for which the conjecture of Effros and Shen is false. If $p = r - 1$ holds then we can use the following criterion in order to decide whether G is a simple dimension group or not. Let b be a nonzero vector which is orthogonal to the hyperplane in $\mathbf{R}^r$ which is generated by the vectors $a^{(1)}, \ldots, a^{(r-1)}$.

1.1. PROPOSITION [2, COROLLARY 4.2]. *The following statements are equivalent.*

(1.1.1) *G is a simple dimension group.*

(1.1.2) $\det(a^{(1)}, \ldots, a^{(r-1)}, z) \neq 0$ *holds for any nonzero vector z in $\mathbf{Z}^r$.*

(1.1.3) *The components of the vector b are linearly independent over the rational field* $\mathbf{Q}$.

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2. Henceforth we fix $r \geqslant 3$. Let $a^{(1)}, \ldots, a^{(r-1)}$ be fixed linearly independent vectors in $(\mathbf{R}^r)^+$ such that the ordered group G defined above is a simple dimension group. Denote by b the unique vector in $\mathbf{R}^r$ such that

$$\langle b, x\rangle = \det(a^{(1)}, \ldots, a^{(r-1)}, x).$$

By multiplying one of the vectors $a^{(1)}, \ldots, a^{(r-1)}$ with a suitable constant we may assume that the last component b_r of b is equal to 1. Of course, this does not impair the definition of G.

We need some other notations. For any subset $M \subseteq \mathbf{R}^r$ we denote by $\operatorname{conv}(M)$ the convex hull of M. For any vector $x = (x_1, \ldots, x_r)^t \in \mathbf{R}^r \setminus \{0\}$ we set $\bar{x} = \|x\|^{-1}x$, where $\|x\|$ is the l^1-norm of x, and we set $\check{x} = (x_1, \ldots, x_{r-1})^t$.

Now we assume that $A_1, A_2, \ldots$ is a sequence in $\mathrm{GL}(r, \mathbf{Z})^+$ such that

$$G^+ = \bigcup_{n=1}^{\infty} (A_n \ldots A_1)^{-1}(\mathbf{Z}^r)^+,$$

or equivalently,

$$\operatorname{conv}(\{\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}\}) = \bigcap_{n=1}^{\infty} \{\bar{x}/x \in \operatorname{conv}(A_1^t \ldots A_n^t(\mathbf{Z}^r)^+) \setminus \{0\}\}. \tag{2.1}$$

2.1. Definition [5, II, §4]. To each vector $x \in \mathbf{R}^{r-1}$ we associate a linear form L_x on $\mathbf{Z}^{r-1}$ by $L_x(z) = \langle x, z\rangle$, $z \in \mathbf{Z}^{r-1}$. L_x is called badly approximable if there exists a positive constant α such that $|L_x(z) - q| \geqslant \alpha\|z\|_\infty^{-r+1}$ holds for each $z \in \mathbf{Z}^{r-1} \setminus \{0\}$, $q \in \mathbf{Z}$, where $\|z\|_\infty$ denotes the maximum norm of z.

Our main purpose in this section is to show that in our present situation the linear form $L_{\check{b}}$, as defined above, can not be badly approximable.

We need the following lemma whose easy proof is left to the reader.

2.2. Lemma. *Let $\Delta_1, \Delta_2, \ldots$ be a monotonely decreasing sequence of $r-1$ dimensional simplices in $\mathbf{R}^r$ and let $\Delta = \bigcap_{n=1}^{\infty} \Delta_n$. Moreover let $v^{(1,n)}, \ldots, v^{(r,n)}$ be the extreme points of Δ_n. Then there exists a monotonely increasing sequence $n_1, n_2, \ldots$ of positive integers such that $\{v^{(j,n_i)}\}_{i\in\mathbf{N}}$ converges to a point in Δ for each j, $1 \leqslant j \leqslant r$, and for any extreme point w in Δ there exists a j, $1 \leqslant j \leqslant r$, such that $w = \lim_{i\to\infty} v^{(j,n_i)}$.*

It follows from (2.1) together with 2.2 that there exists a monotonely increasing sequence $n_1, n_2, \ldots$ of positive integers, a permutation σ of the set of integers $\{1, \ldots, r\}$, and a point w in $\operatorname{conv}(\{\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}\})$ such that, if $v^{(\sigma(1),i)}, \ldots, v^{(\sigma(r),i)}$ are the column vectors of the matrix $B^{(i)} = A_1^t \ldots A_{n_i}^t$, then we have

$$\lim_{i\to\infty} \bar{v}^{(j,i)} = \bar{a}^{(i)} \quad \text{for } 1 \leqslant j \leqslant r-1; \qquad \lim_{i\to\infty} \bar{v}^{(r,i)} = w.$$

Now we can prove the following proposition which is crucial for the proof of Theorem 2.4.

2.3. Proposition. *There exists a $k \in \{1, \ldots, r-1\}$ such that*

$$\lim_{n\to\infty} \det(\bar{v}^{(1,n)}, \ldots, \bar{v}^{(r,n)})^{-1} \det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, \bar{v}^{(k,n)}) = 0.$$

PROOF. There exists a $k \in \{1, \ldots, r-1\}$ such that $\bar{a}^{(k)} \neq w$. Since $\operatorname{conv}(\{\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}\})$ is a simplex containing w, we have

$$\bar{a}^{(k)} \notin M = \operatorname{conv}(\{\bar{a}^{(j)}/j \neq k\} \cup \{w\}).$$

It follows that

$$\delta = \inf\{\|\bar{a}^{(k)} - x\|/x \in M\} > 0.$$

Since $\bar{a}^{(k)}$ is contained in $\operatorname{conv}(\{\bar{v}^{(1,n)}, \ldots, \bar{v}^{(r,n)}\})$ for each $n \in \mathbf{N}$, we can write

$$\bar{a}^{(k)} = \sum_{j=1}^{r} t_j^{(n)} \bar{v}^{(j,n)},$$

with $t_j^{(n)} \in [0, 1]$ and $\sum_{j=1}^{r} t_j^{(n)} = 1$. By our assumption G is a simple dimension group. Therefore we obtain from (1.1.2) that $\det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, \bar{v}^{(k,n)}) \neq 0$. In particular $\bar{v}^{(k,n)} \neq \bar{a}^{(k)}$ and $t_k^{(n)} \neq 1$. Thus we can define

$$\lambda_j^{(n)} = t_j^{(n)}(1 - t_k^{(n)})^{-1}, \qquad u^{(n)} = \sum_{j \neq k} \lambda_j^{(n)} \bar{v}^{(j,n)}.$$

Since $\{\bar{v}^{(j,n)}\}_{n \in \mathbf{N}}$ converges to $\bar{a}^{(j)}$ for $1 \leq j \leq r-1$, and to w for $j = r$, there exists a $n_0 \in \mathbf{N}$ such that for any $n \geq n_0$

$$\|\bar{v}^{(j,n)} - \bar{a}^{(j)}\| \leq \delta/2 \quad \text{for } j \neq r; \qquad \|\bar{v}^{(r,n)} - w\| \leq \delta/2.$$

If we set

$$\tilde{u}^{(n)} = \sum_{j \neq k,r} \lambda_j^{(n)} \bar{a}^{(j)} + \lambda_r^{(n)} w$$

then we obtain for each $n \geq n_0$

$$\|u^{(n)} - \tilde{u}^{(n)}\| \leq \delta/2.$$

Since $\tilde{u}^{(n)}$ is contained in M we have

$$\|\tilde{u}^{(n)} - \bar{a}^{(k)}\| \geq \delta.$$

By combining the last two inequalities we obtain an estimation for the distance of $u^{(n)}$ and $\bar{a}^{(k)}$:

$$\|u^{(n)} - \bar{a}^{(k)}\| \geq \delta/2 \quad \text{for } n \geq n_0.$$

Since $\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, u^{(n)}$ are contained in $\operatorname{conv}(\{\bar{v}^{(1,n)}, \ldots, \bar{v}^{(1,r)}\})$ we have

$$|\det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, u^{(n)})| \leq |\det(\bar{v}^{(1,n)}, \ldots, \bar{v}^{(r,n)})|.$$

Therefore we obtain for each $n \geq n_0$

$$\frac{|\det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, \bar{v}^{(k,n)})|}{|\det(\bar{v}^{(1,n)}, \ldots, \bar{v}^{(r,n)})|}$$

$$\leq \frac{|\det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, t_k^{(n)^{-1}}(\bar{a}^{(k)} - u^{(n)}) + u^{(n)})|}{|\det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, u^{(n)})|} = |1 - t_k^{(n)^{-1}}|.$$

Since $\|u^{(n)} - \bar{a}^{(k)}\| \geq \delta/2$ holds for each $n \geq n_0$ and since

$$\|\bar{a}^{(k)} - \bar{v}^{(k,n)}\| = |1 - t_k^{(n)^{-1}}| \, \|u^{(n)} - \bar{a}^{(k)}\|$$

converges to zero for $n \to \infty$ it follows that $|1 - t_k^{(n)^{-1}}|$ converges to zero. Hence

$$\lim_{n\to\infty} \det(\bar{v}^{(1,n)}, \ldots, \bar{v}^{(r,n)})^{-1} \det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, \bar{v}^{(k,n)}) = 0.$$

2.4. THEOREM. *In the situation considered above the linear form $L_{\check{b}}$ is not badly approximable.*

PROOF. Suppose that our assertion is not true. Then there exists a positive constant α such that

$$|\langle b, z\rangle| \geqslant \alpha \|\check{z}\|_\infty^{-r+1} \quad \text{for each } z \in \mathbf{Z}^r \setminus \{0\}. \tag{2.4.1}$$

First we show the following.

(2.4.2) There exists a positive constant β such that

$$\|v^{(i,n)}\| \, \|v^{(j,n)}\|^{-1} \leqslant \beta \quad \text{for each } n \in \mathbf{N}; \qquad i, j \in \{1, \ldots, r\}.$$

Suppose that this is not true. Let $k_n = \min\{\|v^{(j,n)}\|/1 \leqslant j \leqslant r\}$ and $l_n = \|v^{(1,n)}\| \, \|v^{(2,n)}\| \cdots \|v^{(r,n)}\|$. Then there exists a monotonely increasing sequence $n_1, n_2, \ldots$ of positive integers such that $\lim_{i\to\infty} k_{n_i}^r l_{n_i}^{-1} = 0$. We can find an $m \in \{1, \ldots, r\}$ such that $\|v^{(m,n_i)}\| = k_{n_i}$ for infinitely many $i \in \mathbf{N}$. Thus we may choose the sequence $n_1, n_2, \ldots$ in such a manner that in addition $\|v^{(m,n_i)}\| = k_{n_i}$ holds for each $i \in \mathbf{N}$. We set $d = \|a^{(1)}\| \cdots \|a^{(r-1)}\|$. Now we obtain from the inequality

$$|\det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, \bar{v}^{(m,n)})| \leqslant |\det(\bar{v}^{(1,n)}, \ldots, \bar{v}^{(r,n)})|,$$

which is true for any $n \in \mathbf{N}$,

$$\begin{aligned} k_{n_i}^{r-1}|\det(a^{(1)}, \ldots, a^{(r-1)}, v^{(m,n_i)})| &= k_{n_i}^r d^{-1}|\det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, \bar{v}^{(m,n_i)})| \\ &\leqslant k_{n_i}^r d^{-1}|\det(\bar{v}^{(1,n_i)}, \ldots, \bar{v}^{(r,n_i)})| \\ &= k_{n_i}^r l_{n_i}^{-1} d^{-1}|\det(v^{(1,n_i)}, \ldots, v^{(r,n_i)})| = k_{n_i}^r l_{n_i}^{-1} d^{-1}. \end{aligned}$$

From this we infer that

$$\lim_{i\to\infty} \|v^{(m,n_i)}\|^{r-1}|\det(a^{(1)}, \ldots, a^{(r-1)}, v^{(m,n_i)})| = 0.$$

Since $\|\check{v}^{(m,n_i)}\|_\infty \leqslant \|v^{(m,n_i)}\|$ holds for each $i \in \mathbf{N}$, this implies

$$\lim_{i\to\infty} \|\check{v}^{(m,n_i)}\|_\infty^{r-1}|\langle b, v^{(m,n_i)}\rangle| = 0.$$

However, this contradicts (2.4.1).

Using 2.3 as well as (2.4.2) we can now easily complete the proof of our theorem. Let $k \in \{1, \ldots, r-1\}$ be chosen as in 2.3. By (2.4.2) there exists a positive constant β such that

$$\|v^{(k,n)}\|^r \leqslant \beta \|v^{(1,n)}\| \cdots \|v^{(r,n)}\| \quad \text{for any } n \in \mathbf{N}.$$

Since $|\det(B_n)| = 1$ holds for each $n \in \mathbf{N}$, we obtain now

$$\begin{aligned} \beta|\det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, \bar{v}^{(k,n)})| \, |\det(\bar{v}^{(1,n)}, \ldots, \bar{v}^{(r,n)})|^{-1} \\ \geqslant \|v^{(k,n)}\|^{r-1}|\det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, v^{(k,n)})| \\ = \|v^{(k,n)}\|^{r-1} d^{-1}|\det(a^{(1)}, \ldots, a^{(r-1)}, v^{(k,n)})|. \end{aligned}$$

Therefore, by 2.3,

$$\lim_{n\to\infty} \|\check{v}^{(k,n)}\|_\infty^{r-1} |\langle b, v^{(k,n)}\rangle| = 0,$$

and this contradicts (2.4.1).

3. By using the results of §2 we are now able to construct a lot of counterexamples to the unimodular conjecture. Let x be a vector in $\mathbf{R}^{r-1}$ such that L_x is badly approximable. By [5, II, §4, Theorem 4A] we may choose x in such a manner that the components $x_1, \ldots, x_{r-1}$ of x lie in an algebraic number field K of degree r and $\{1, x_1, \ldots, x_{r-1}\}$ is a basis of K. We also demand that the components of x do not all have a positive sign. (If necessary, change the basis in $\mathbf{Z}^{r-1}$.) We set $b = (x_1, \ldots, x_{r-1})^t$, and we choose $r-1$ linearly independent positive vectors in the hyperplane which is orthogonal to b. (Since the components of x do not all have a positive sign this can always be done.) Now we define a simple dimension group G as in §1. In order to show that G cannot be obtained as the inductive limit of a unimodular system of groups it is necessary to state that the property that b gives rise to a badly approximable linear form L_b is left invariant under isomorphisms of the group G. To speak more precisely, if G' is an isomorphic copy of G and we associate a vector b' with G' in the same manner as the vector b with G, then we have $\gamma b' = Ab$ for some $A \in \mathrm{GL}(r, \mathbf{Z})$ and a nonzero constant γ. Now some simple calculations show that $L_{b'}$ is also badly approximable. Thus, by an application of 2.4, we can conclude that G is not isomorphic to the inductive limit of a system $\mathbf{Z}^r \xrightarrow{A_1} \mathbf{Z}^r \xrightarrow{A_2} \cdots \mathbf{Z}^r \xrightarrow{A_n} \ldots$ with $A_n \in \mathrm{GL}(r, \mathbf{Z})^+$ for each $n \in \mathbf{N}$.

It follows from a theorem of Khintchine (see [5, III, §3, Theorem 3A]) together with a transference principle for badly approximable linear forms (see [5, IV, §5, Theorem 5B]) that the set of all vectors $x \in \mathbf{R}^{r-1}$ such that L_x is badly approximable has Lebesgue measure zero. Thus there is still some hope that the conjecture of Effros and Shen is valid for a rather big class of finitely generated dimension groups.

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