A COUNTEREXAMPLE TO THE UNIMODULAR CONJECTURE ON FINITELY GENERATED DIMENSION GROUPS

NORBERT RIEDEL

Abstract. We give a series of examples of simple finitely generated dimension groups which cannot be obtained as the inductive limit of a system

\[ \mathbb{Z}^{A_1} \to \mathbb{Z}^{A_2} \to \cdots \to \mathbb{Z}^{A_n} \to \cdots, \]

where each \( A_n \) is a unimodular matrix whose entries are nonnegative integers.

1. In this note we are concerned with ordered groups \( G \) of the following form. \( G \) is equal to \( \mathbb{Z}^r \) as an abelian group only, for some \( r \in \mathbb{N} \), and there exists a set \( \{a(1), \ldots, a(r)\} \) of linearly independent vectors in \((\mathbb{R}^r)^+\) such that the positive cone \( G^+ \) of \( G \) is given by

\[ G^+ = \left\{ z \in \mathbb{Z}^r / \langle a(i), z \rangle > 0; i = 1, \ldots, r \right\} \cup \{0\}. \]

\( G \) is a dimension group if it satisfies the Riesz interpolation property (for the definitions concerning dimension groups we refer to [1], [2]). Effros and Shen conjectured in [3] (see also [1]) that if \( G \) is a dimension group, then there exists a sequence \( A_1, A_2, \ldots \) in the set \( \text{GL}(r, \mathbb{Z})^+ \) of all unimodular matrices whose entries are nonnegative integers, such that

\[ G^+ = \bigcup_{n=1}^{\infty} (A_n \cdots A_1)^{-1}(\mathbb{Z}^r)^+. \]

In [4] we have shown that the conjecture is true if \( G \) is simple (i.e. \( G^+ \) has no nontrivial faces) and \( p = 1 \) holds. Using the theory of diophantine approximation we will show in the sequel that for \( p = r - 1 \) (\( r > 3 \)) there exist simple dimension groups for which the conjecture of Effros and Shen is false. If \( p = r - 1 \) holds then we can use the following criterion in order to decide whether \( G \) is a simple dimension group or not. Let \( b \) be a nonzero vector which is orthogonal to the hyperplane in \( \mathbb{R}^r \) which is generated by the vectors \( a(1), \ldots, a(r-1) \).

1.1. Proposition [2, Corollary 4.2]. The following statements are equivalent.

1.1.1 \( G \) is a simple dimension group.

1.1.2 \( \det(a(1), \ldots, a(r-1), z) \neq 0 \) holds for any nonzero vector \( z \) in \( \mathbb{Z}^r \).

1.1.3 The components of the vector \( b \) are linearly independent over the rational field \( \mathbb{Q} \).

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2. Henceforth we fix $r > 3$. Let $a^{(1)}, \ldots, a^{(r-1)}$ be fixed linearly independent vectors in $(\mathbb{R}^r)^\ast$ such that the ordered group $G$ defined above is a simple dimension group. Denote by $b$ the unique vector in $\mathbb{R}^r$ such that 

$$<b, x> = \det(a^{(1)}, \ldots, a^{(r-1)}, x).$$

By multiplying one of the vectors $a^{(1)}, \ldots, a^{(r-1)}$ with a suitable constant we may assume that the last component $b_r$ of $b$ is equal to 1. Of course, this does not impair the definition of $G$.

We need some other notations. For any subset $M \subseteq \mathbb{R}^r$ we denote by $\text{conv}(M)$ the convex hull of $M$. For any vector $x = (x_1, \ldots, x_r)' \in \mathbb{R}^r \setminus \{0\}$ we set $\bar{x} = ||x||^{-1} x$, where $||x||$ is the $l^1$-norm of $x$, and we set $\bar{x} = (x_1, \ldots, x_{r-1})'$.

Now we assume that $A_1, A_2, \ldots$ is a sequence in $\text{GL}(r, \mathbb{Z})^\ast$ such that 

$$G^+ = \bigcup_{n=1}^{\infty} (A_n \ldots A_1)^{-1}(\mathbb{Z}^r)^\ast,$$

or equivalently,

$$\text{conv}\{\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}\} = \bigcap_{n=1}^{\infty} \{\bar{x} \in \text{conv}(A_n' \ldots A_1')(\mathbb{Z}^r)^\ast \setminus \{0\}\}. \quad (2.1)$$

2.1. Definition [5, II, §4]. To each vector $x \in \mathbb{R}^{r-1}$ we associate a linear form $L_x$ on $\mathbb{Z}^{r-1}$ by $L_x(z) = <x, z>$, $z \in \mathbb{Z}^{r-1}$. $L_x$ is called badly approximate if there exists a positive constant $\alpha$ such that $|L_x(z) - q| > \alpha||z||_\infty^{-r+1}$ holds for each $z \in \mathbb{Z}^{r-1} \setminus \{0\}$, $q \in \mathbb{Z}$, where $||z||_\infty$ denotes the maximum norm of $z$.

Our main purpose in this section is to show that in our present situation the linear form $L_x$, as defined above, cannot be badly approximable.

We need the following lemma whose easy proof is left to the reader.

2.2. Lemma. Let $A_1, A_2, \ldots$ be a monotonely decreasing sequence of $r-1$ dimensional simplices in $\mathbb{R}^r$ and let $A = \bigcap_{n=1}^{\infty} A_n$. Moreover let $v^{(1,0)}, \ldots, v^{(r,0)}$ be the extreme points of $A_n$. Then there exists a monotonely increasing sequence $n_1, n_2, \ldots$ of positive integers such that $(v^{(j, n)})_{i \in \mathbb{N}}$ converges to a point in $A$ for each $j$, $1 \leq j \leq r$, and for any extreme point $w$ in $A$ there exists a $j$, $1 \leq j \leq r$, such that $w = \lim_{i \rightarrow \infty} v^{(j, n)}$.

It follows from (2.1) together with 2.2 that there exists a monotonely increasing sequence $n_1, n_2, \ldots$ of positive integers, a permutation $\sigma$ of the set of integers $\{1, \ldots, r\}$, and a point $w$ in $\text{conv}(\{\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}\})$ such that, if $v^{(6(1),0)}, \ldots, v^{(6(r),0)}$ are the column vectors of the matrix $B^{(i)} = A^{(i)}_1 \ldots A^{(i)}_n$, then we have 

$$\lim_{i \rightarrow \infty} v^{(j, 0)} = \bar{a}^{(i)} \quad \text{for } 1 < j < r - 1; \quad \lim_{i \rightarrow \infty} v^{(r, 0)} = w.$$ 

Now we can prove the following proposition which is crucial for the proof of Theorem 2.4.

2.3. Proposition. There exists a $k \in \{1, \ldots, r-1\}$ such that 

$$\lim_{n \rightarrow \infty} \det(v^{(1,n)}, \ldots, v^{(r,n)})^{-1} \det(\bar{a}^{(1)}, \ldots, \bar{a}^{(r-1)}, \bar{v}^{(k,n)}) = 0.$$
Proof. There exists a \( k \in \{1, \ldots, r - 1\} \) such that \( \tilde{a}^{(k)} \neq w \). Since \( \text{conv}(\{ \tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)} \}) \) is a simplex containing \( w \), we have
\[
\tilde{a}^{(k)} \notin M = \text{conv}(\{ \tilde{a}^{(j)} / j \neq k \} \cup \{ w \}).
\]

It follows that
\[
\delta = \inf \{ \| \tilde{a}^{(k)} - x \| / x \in M \} > 0.
\]
Since \( \tilde{a}^{(k)} \) is contained in \( \text{conv}(\{ \tilde{v}^{(1,n)}, \ldots, \tilde{v}^{(r,n)} \}) \) for each \( n \in \mathbb{N} \), we can write
\[
\tilde{a}^{(k)} = \sum_{j=1}^{r} f_j^{(n)} \tilde{v}^{(j,n)},
\]
with \( f_j^{(n)} \in [0, 1] \) and \( \sum_{j=1}^{r} f_j^{(n)} = 1 \). By our assumption \( G \) is a simple dimension group. Therefore we obtain from (1.1.2) that \( \det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, \tilde{v}^{(k,n)}) \neq 0 \). In particular \( \tilde{v}^{(k,n)} \neq \tilde{a}^{(k)} \) and \( f_k^{(n)} \neq 1 \). Thus we can define
\[
\lambda_j^{(n)} = f_j^{(n)} (1 - f_k^{(n)})^{-1}, \quad u^{(n)} = \sum_{j \neq k} \lambda_j^{(n)} \tilde{v}^{(j,n)}.
\]
Since \( \{ \tilde{v}^{(j,n)} \}_{n \in \mathbb{N}} \) converges to \( \tilde{a}^{(j)} \) for \( 1 < j < r - 1 \), and to \( w \) for \( j = r \), there exists a \( n_0 \in \mathbb{N} \) such that for any \( n > n_0 \)
\[
\| \tilde{v}^{(j,n)} - \tilde{a}^{(j)} \| < \delta/2 \quad \text{for } j \neq r; \quad \| \tilde{v}^{(r,n)} - w \| < \delta/2.
\]

If we set
\[
\tilde{u}^{(n)} = \sum_{j \neq k, r} \lambda_j^{(n)} \tilde{a}^{(j)} + \lambda_r^{(n)} w
\]
then we obtain for each \( n > n_0 \)
\[
\| u^{(n)} - \tilde{a}^{(k)} \| < \delta/2.
\]
Since \( \tilde{u}^{(n)} \) is contained in \( M \) we have
\[
\| \tilde{u}^{(n)} - \tilde{a}^{(k)} \| > \delta.
\]
By combining the last two inequalities we obtain an estimation for the distance of \( u^{(n)} \) and \( \tilde{a}^{(k)} \):
\[
\| u^{(n)} - \tilde{a}^{(k)} \| > \delta/2 \quad \text{for } n > n_0.
\]
Since \( \tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, u^{(n)} \) are contained in \( \text{conv}(\{ \tilde{v}^{(1,n)}, \ldots, \tilde{v}^{(r,n)} \}) \) we have
\[
|\det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, u^{(n)})| < |\det(\tilde{v}^{(1,n)}, \ldots, \tilde{v}^{(r,n)})|.
\]

Therefore we obtain for each \( n > n_0 \)
\[
\frac{|\det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, \tilde{v}^{(k,n)})|}{|\det(\tilde{v}^{(1,n)}, \ldots, \tilde{v}^{(r,n)})|} \leq \frac{|\det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, f_k^{(n)-1}(\tilde{a}^{(k)} - u^{(n)}) + u^{(n)})|}{|\det(\tilde{a}^{(1)}, \ldots, \tilde{a}^{(r-1)}, u^{(n)})|} = |1 - f_k^{(n)}|^{-1}.
\]

Since \( \| u^{(n)} - \tilde{a}^{(k)} \| > \delta/2 \) holds for each \( n > n_0 \) and since
\[
\| \tilde{a}^{(k)} - \tilde{v}^{(k,n)} \| = |1 - f_k^{(n)}| \quad \| u^{(n)} - \tilde{a}^{(k)} \|
\]
converges to zero for $n \to \infty$ it follows that $|1 - r_k^{(n)}|^{-1}$ converges to zero. Hence

$$
\lim_{n \to \infty} \det(\tilde{v}(1,n), \ldots, \tilde{v}(r,n))^{-1} \det(\tilde{a}(1), \ldots, \tilde{a}(r-1), \tilde{v}(k,n)) = 0.
$$

2.4. **Theorem.** In the situation considered above the linear form $L_k$ is not badly approximable.

**Proof.** Suppose that our assertion is not true. Then there exists a positive constant $\alpha$ such that

$$
|\langle b, z \rangle| > \alpha \|z\|^{-r+1}
$$

for each $z \in \mathbb{Z} \setminus \{0\}$. (2.4.1)

First we show the following.

(2.4.2) There exists a positive constant $\beta$ such that

$$
\|v^{(i,n)}\| \|v^{(j,n)}\|^{-1} < \beta \quad \text{for each } n \in \mathbb{N}; \quad i, j \in \{1, \ldots, r\}.
$$

Suppose that this is not true. Let $k_n = \min\{|v^{(j,n)}|/1 < j < r\}$ and $l_n = \|v^{(1,n)}\| \|v^{(2,n)}\| \cdots \|v^{(r,n)}\|$. Then there exists a monotonically increasing sequence $n_1, n_2, \ldots$ of positive integers such that $\lim_{n \to \infty} k_n^{r-1} = 0$. We can find an $m \in \{1, \ldots, r\}$ such that $\|v^{(m,n)}\| = k_n$ for infinitely many $n \in \mathbb{N}$. Thus we may choose the sequence $n_1, n_2, \ldots$ in such a manner that in addition $\|v^{(m,n)}\| = k_n$ holds for each $n \in \mathbb{N}$. We set $d = \|a^{(1)}\| \cdots \|a^{(r-1)}\|$. Now we obtain from the inequality

$$
|\det(\tilde{a}(1), \ldots, \tilde{a}(r-1), v^{(m,n)})| \leq |\det(\tilde{v}(1,n), \ldots, \tilde{v}(r,n))|,
$$

which is true for any $n \in \mathbb{N},$

$$
k_n^{r-1}|\det(\tilde{a}(1), \ldots, \tilde{a}(r-1), v^{(m,n)})| = k_n^{r-1}d^{-1}|\det(\tilde{a}(1), \ldots, \tilde{a}(r-1), v^{(m,n)})|

< k_n^{r-1}d^{-1}|\det(\tilde{v}(1,n), \ldots, \tilde{v}(r,n))|,

= k_n^{r-1}d^{-1}|\det(\tilde{v}(1,n), \ldots, \tilde{v}(r,n))| = k_n^{r-1}d^{-1}.
$$

From this we infer that

$$
\lim_{i \to \infty} \|v^{(m,n)}\|^{r-1}|\det(\tilde{a}(1), \ldots, \tilde{a}(r-1), v^{(m,n)})| = 0.
$$

Since $\|v^{(m,n)}\| \to \infty < \|v^{(m,n)}\|$ holds for each $i \in \mathbb{N}$, this implies

$$
\lim_{i \to \infty} \|v^{(m,n)}\|^{r-1}|\langle b, v^{(m,n)} \rangle| = 0.
$$

However, this contradicts (2.4.1).

Using 2.3 as well as (2.4.2) we can now easily complete the proof of our theorem. Let $k \in \{1, \ldots, r-1\}$ be chosen as in 2.3. By (2.4.2) there exists a positive constant $\beta$ such that

$$
\|v^{(k,n)}\| < \beta \|v^{(1,n)}\| \cdots \|v^{(r,n)}\| \quad \text{for any } n \in \mathbb{N}.
$$

Since $|\det(B_n)| = 1$ holds for each $n \in \mathbb{N},$ we obtain now

$$
\beta |\det(\tilde{a}(1), \ldots, \tilde{a}(r-1), v^{(k,n)})| |\det(\tilde{v}(1,n), \ldots, \tilde{v}(r,n))|^{-1}

> \|v^{(k,n)}\|^{r-1}|\det(\tilde{a}(1), \ldots, \tilde{a}(r-1), v^{(k,n)})|

= \|v^{(k,n)}\|^{r-1}|\det(\tilde{a}(1), \ldots, \tilde{a}(r-1), v^{(k,n)})|.
$$
Therefore, by 2.3,
\[
\lim_{n \to \infty} \|\tilde{v}^{(k,n)}\|_{\infty} (r-1) \langle b, v^{(k,n)} \rangle = 0,
\]
and this contradicts (2.4.1).

3. By using the results of §2 we are now able to construct a lot of counterexamples to the unimodular conjecture. Let \( x \) be a vector in \( \mathbb{R}^{r-1} \) such that \( L_x \) is badly approximable. By [5, II, §4, Theorem 4A] we may choose \( x \) in such a manner that the components \( x_1, \ldots, x_{r-1} \) of \( x \) lie in an algebraic number field \( K \) of degree \( r \) and \( \{1, x_1, \ldots, x_{r-1}\} \) is a basis of \( K \). We also demand that the components of \( x \) do not all have a positive sign. (If necessary, change the basis in \( \mathbb{Z}^{r-1} \).) We set \( b = (x_1, \ldots, x_{r-1})' \), and we choose \( r - 1 \) linearly independent positive vectors in the hyperplane which is orthogonal to \( b \). (Since the components of \( x \) do not all have a positive sign this can always be done.) Now we define a simple dimension group \( G \) as in §1. In order to show that \( G \) cannot be obtained as the inductive limit of a unimodular system of groups it is necessary to state that the property that \( b \) gives rise to a badly approximable linear form \( L_b \) is left invariant under isomorphisms of the group \( G \). To speak more precisely, if \( G' \) is an isomorphic copy of \( G \) and we associate a vector \( b' \) with \( G' \) in the same manner as the vector \( b \) with \( G \), then we have \( \gamma b' = Ab \) for some \( A \in \text{GL}(r, \mathbb{Z}) \) and a nonzero constant \( \gamma \). Now some simple calculations show that \( L_{b'} \) is also badly approximable. Thus, by an application of 2.4, we can conclude that \( G \) is not isomorphic to the inductive limit of a system \( \mathbb{Z}^r \rightarrow \mathbb{Z}^r \rightarrow \cdots \mathbb{Z}^r \rightarrow \cdots \) with \( A_n \in \text{GL}(r, \mathbb{Z})^+ \) for each \( n \in \mathbb{N} \).

It follows from a theorem of Khintchine (see [5, III, §3, Theorem 3A]) together with a transference principle for badly approximable linear forms (see [5, IV, §5, Theorem 5B]) that the set of all vectors \( x \in \mathbb{R}^{r-1} \) such that \( L_x \) is badly approximable has Lebesgue measure zero. Thus there is still some hope that the conjecture of Effros and Shen is valid for a rather big class of finitely generated dimension groups.

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Institut für Mathematik, Technische Universität, D-8000 München, Federal Republic of Germany

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