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ON THE SUPERCENTER OF A GROUP OVER DOMAINS OF CHARACTERISTIC 0

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ABSTRACT. Let G be a group and R a commutative ring with 1 and let $U_R G$ be the group of units of the group ring RG. The R-supercenter of G, $S_R G$ is the set of elements of G which have a finite number of conjugates under $U_R G$. The supercenter is studied in the case where R is a domain of characteristic 0. It is shown that for the most important cases the supercenter coincides with the center of the group and with the intersection of all group bases of the group ring.

1. Introduction. We use the following notations:

G is a group, TG the torsion of G, $FC(G)$ the FC-part of G;

R is a commutative ring with 1; $U_R G$ (resp. $V_R G$) is the group of units (resp. normalized units) of the group ring RG;

$FC_R G$ is the FC-part of $V_R G$;

$S_R G$, the R-supercenter of G, is the intersection of G with $FC_R G$, i.e. the set of elements of G which have a finite number of conjugates under $U_R G$.

We wish to investigate here the supercenter, $S_R G$, in the case where R is an integral domain of characteristic 0. The integral case, $S_Z G$, has been studied in **[5]** and the case of infinite fields, $S_K G$, in **[3]**. For the proofs we use some techniques, different from the ones used in those papers, which were introduced in **[1]** and **[2]**. Our Theorem 1 is a generalization of Theorem A of **[3]** (see Corollary 1).

Our results are the following:

THEOREM 1. *Let G be a torsion group.*

(A) *If R has characteristic* 0, *if* $\{|g| \mid g \in G\} \cap U(R) = \{1\}$ *and if the Jacobson radical of R is different from* 0, *then* $TFC_R G = S_R G = Z(G)$.

(B) *If R contains a subring R_1 which is a domain of characteristic* 0 *with nonzero Jacobson radical, then* $S_R G = Z(G)$.

COROLLARY 1. *The supercenter of a torsion group over a field of characteristic* 0 *is equal to the center of the group.*

THEOREM 2.

(A) *Let R be an integral domain such that no rational prime is a unit of R. Then,* $TS_R G = TS$, *where S is the intersection of all group bases of RG.*

(B) *Let G be a torsion group and let R be an integral domain of characteristic* 0 *such that* $\{|g| \mid g \in G\} \cap U(R) = \{1\}$. *Then* $S_R G = S$.

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REMARK. It is easy to see that, for either case (A) or (B) in the theorems, the torsion of the R-supercenter of G is normal in $U_R G$.

2. Proofs.

Theorem 1, (A). It suffices to show that if x is a torsion element in $FC_R G$ then x is in the center $Z(G)$ of G. We proceed by contradiction assuming that there is an element t in G, of order m, which does not commute with x.

Since the support of x is in $FC(G)$, this set together with t generate a finite group. Hence, we can assume that G is finite.

Let $r \neq 0$ be an element in the radical of R and let us consider the unit $u = \sum_{i=0}^{m-1} r^i t^i$ with $u^{-1} = (1 - rt)(1 - r^m)^{-1}$. Then we must have that $[u, x]$ is a unit of finite order but this is a contradiction because the coefficient of 1 in $[u, x]$ is different from 0.

Theorem 1, (B). We reduce this to the previous case. We proceed as above and pick r in the radical of R_1 such that either r is integral over Z or transcendent over Q. We require also that r is a nonzero multiple of n, the order of $G = \langle x, t\rangle$.

If D is the multiplicative subset formed by the elements of $Z[r]$ which are congruent to 1 modulo r, we introduce the ring $R' = D^{-1}(Z[r]) \subset R$.

In order that the proof for part (A) be applicable here, we only have to show that no proper divisor d of n is a unit of R'. But, since d^{-1} is of the form $P(r)/(1 + Q(r)r)$, where P, Q are polynomials with coefficients in Z, we deduce that, if d is invertible in R', then it is invertible in $Z[r]$ and, therefore, $d = 1$.

Theorem 2, *proof that* $TS \subset TS_R G$. For this part we do not need the stronger assumption about the units of R.

It suffices to show that if, for some prime p, s is a p-element of S, then s has a finite number of conjugates under $U_R G$. It is clear that these conjugates are also in TS and then the lemma below says that they are obtained by conjugation by G. Hence, it will be enough to prove that for all g in G, $gsg^{-1} \in \langle s\rangle$.

If $v = (1 + s + \cdots + s^{|s|-1}) \cdot g \cdot (1 - s)$ we see that $u = 1 + v$ is a unit with inverse $1 - v$ and, analyzing the relation $usu^{-1} \in TS \subset G$, the claim follows.

LEMMA. *Let R have characteristic 0 and assume that p is a rational prime which is not invertible in R. If s, t are p-elements of G which are conjugate under a unit u of $U_R G$, then they are also conjugate under an element of G.*

PROOF. We proceed by contradiction, assuming that s, t are not conjugate in G. Writing u in the form $\sum \alpha_g g$, we have

$$\sum \alpha_g gs = \sum \alpha_g tg. \tag{$*$}$$

If M_α is the set of elements g of G which appear in $(*)$ with coefficient $\alpha \neq 0$, we see that $(*)$ defines a permutation σ, of M_α, given by the identities $gs = t\sigma(g)$. If Γ, of length k, is an orbit of M_α under $\langle\sigma\rangle$, $\Gamma = \{g, \sigma(g), \ldots, \sigma^{k-1}(g)\}$, we have

$$gs = t\sigma(g), \qquad \sigma(g)s = t\sigma^2(g), \ldots, \sigma^{k-1}(g)s = tg$$

and it is easy to see that k is the least nonnegative integer such that $s^k = g^{-1}t^k g = (g^{-1}tg)^k$. Our assumption that s, t are not conjugate in G implies then that p divides

α. Since this is true for every α, we get that p divides the augmentation of u, a contradiction because p is not invertible in R.

Theorem 2, *proof that* $TS_RG \subset TS$. Let us consider, as before, a p-element s which now we assume to be a member of S_RG. Calling G an arbitrary group basis of RG, we have to show that $s \in G$. Let us choose t in G as follows: in case (A), we apply [**4**, VI, 2.1] to pick t a p-element in the support of s; in case (B), we choose any t in the support of s. Then, $(s^{-1}t)^{|t|}$ is a product of conjugates of s^{-1} and, therefore, has finite order. Hence, $s^{-1}t$ has finite order. Then, in case (A), we apply [**4**, II, 1.2] and, in case (B) [**4**, II, 1.4], and we obtain the desired conclusion: $s = t$.

Other immediate consequences of our results are

COROLLARY 2. *For either case* (A) *or* (B) *of Theorem* 2, *if N is a torsion, normal subgroup of V_RG, then N is contained in the R-supercenter of G.* (*Use the techniques for the proof of Theorem* 2.)

COROLLARY 3. *Let R contain a domain of characteristic* 0 *with nonzero Jacobson radical. Let G be a torsion group and let us denote by S the intersection of all group bases of RG. Then S is contained in the center of G. If, furthermore, R is a domain of characteristic* 0 *such that* $\{|g| \mid g \in G\} \cap U(R) = 1$, *then* $S = Z(G)$.

PROOF. By the proof of Theorem 1, R contains R' which satisfies the hypotheses of Theorem 2, (B). Hence, $S_RG = Z(G)$ is the intersection of the group bases of $R'G$. Since every group basis of $R'G$ is a group basis of RG, $S \subset Z(G)$. On the other hand, if R has the properties listed at the end, it is obvious that $Z(R) \subset S$.

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