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ON SIMPLE REDUCIBLE LIE ALGEBRAS OF DEPTH TWO

THOMAS B. GREGORY

ABSTRACT. We show that under certain circumstances simple finite-dimensional reducible graded Lie algebras of the form $L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k$ can be given irreducible transitive gradations of the form $M_{-1} \oplus M_0 \oplus \cdots \oplus M_{[k/2]}$.

In [1], the present author classified the simple finite-dimensional irreducible graded Lie algebras over an algebraically closed field of characteristic $p > 5$ which have the form

$$L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k \tag{1}$$

where L_0 is classical and reductive and $k \geqslant 3$. In this paper, we consider certain reducible Lie algebras L of the form (1). In them, we show how to construct an irreducible, transitive gradation

$$L = M_{-1} \oplus M_0 \oplus \cdots \oplus M_{[k/2]}. \tag{2}$$

THEOREM. *Let L be a simple finite-dimensional reducible graded Lie algebra of the form* (1) *over a field of any characteristic, and suppose that L_0 contains no nonzero abelian ideal which annihilates L_{-2} and that $L_{-1} = S + T$, where S and T are proper L_0-submodules of L_{-1}. Then S and T are irreducible abelian L_0-submodules, and $L_{-1} = S \oplus T$. In addition, L possesses an irreducible transitive gradation of the form* (2).

We will prove this theorem by means of a series of lemmas. We begin by restating the first three lemmas of [1]. Their proofs there are valid in this setting, also.

LEMMA 1. *If L is a simple graded Lie algebra of the form* (1), *then L_{-2} is an irreducible L_0-module. Furthermore, $[L_{-1}, L_{-1}] \neq \{0\}$, so $[L_{-1}, L_{-1}] = L_{-2}$; also, $L_{-1} = [L_{-2}, L_1]$. Lastly, $[L_{-1}, x] \neq \{0\}$ for all $x \notin L_{-2}$.*

LEMMA 2. *If L is a simple graded Lie algebra of the form* (1), *then L_k is an irreducible L_0-module, and $L_{j-1} = [L_j, L_{-1}]$, for $-1 \leqslant j \leqslant k$.*

LEMMA 3. *We have $[L_{-2}, L_i] \neq \{0\}$, $0 \leqslant i \leqslant k$, and $[L_0, L_k] \neq \{0\}$.*

LEMMA 4. *L_{-1} contains no proper nonabelian L_0-submodule.*

PROOF. Let N be a nonzero nonabelian L_0-submodule of L_{-1}. Then $[N, N]$ is a nonzero L_0-submodule of the irreducible L_0-submodule L_{-2}, so $[N, N] = L_{-2}$.

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Then we have by Lemma 1 that $L_{-1} = [L_{-2}, L_1] = [[N, N], L_1] \subseteq [[N, L_1], N] + [N, [N, L_1]] \subseteq [L_0, N] + [N, L_0] \subseteq N$, so that $N = L_{-1}$. Q.E.D.

Lemmas 5 through 9 are valid for any simple finite-dimensional L of the form (1) where L_0 contains no nonzero abelian ideal which annihilates L_{-2} and where L_{-1} contains a proper L_0-submodule S.

LEMMA 5. *Let X be an L_0-submodule of L_0 such that $[L_{-1}, X]$ is abelian and $[L_{-2}, X] = \{0\}$. Then $X = \{0\}$.*

PROOF. We have $[L_{-1}, [[L_{-1}, X], X]] \subseteq [[L_{-1}, [L_{-1}, X]], X] + [[L_{-1}, X], [L_{-1}, X]] \subseteq [L_{-2}, X] + \{0\} = \{0\}$. Thus, by Lemma 1, $[[L_{-1}, X], X] = \{0\}$. Then $[L_{-1}, [X, X]] \subseteq [[L_{-1}, X], X] + [X, [L_{-1}, X]] = \{0\}$, so $[X, X] = \{0\}$. Since L_0 is assumed to contain no nonzero abelian ideal which annihilates L_{-2}, we must have $X = \{0\}$. Q.E.D.

LEMMA 6. $[S, [S, L_2]] = \{0\}$.

PROOF. We have that $[L_{-1}, [S, [S, L_2]]] \subseteq [[L_{-1}, S], [S, L_2]] + [S, [[L_{-1}, S], L_2]] + [S, [S, [L_{-1}, L_2]]] \subseteq [S, [[L_{-1}, S], L_2]] + [S, L_0] + [S, L_0] \subseteq S$, which is abelian by Lemma 4. Furthermore, $[L_{-2}, [S, [S, L_2]]] = [S, [S, [L_{-2}, L_2]]] \subseteq [S, [S, L_0]] \subseteq [S, S] = \{0\}$. Thus, by Lemma 5, $[S, [S, L_2]] = \{0\}$. Q.E.D.

LEMMA 7. *S is contained in a maximal L_0-submodule U of L_{-1} such that $U = \mathrm{Ann}_{L_{-1}} U$.*

PROOF (REFEREE). Let U be a maximal proper L_0-submodule of L_{-1} containing S. Then by Lemma 4, $\{0\} = [U, U] = [U, S]$, so $U \subseteq \mathrm{Ann}_{L_{-1}} U \subseteq \mathrm{Ann}_{L_{-1}} S$. Since $[L_{-1}, S] \neq \{0\}$ (by Lemma 1), the maximality of U forces $U = \mathrm{Ann}_{L_{-1}} U = \mathrm{Ann}_{L_{-1}} S$. Q.E.D.

In what follows, we let $S = U$; i.e., we assume that S is a maximal L_0-submodule of L_{-1} which is equal to its own annihilator.

LEMMA 8. $\mathrm{Ann}_{L_1} L_{-2} = \{0\}$.

PROOF. Set $A = \mathrm{Ann}_{L_1} L_{-2}$. Then $[L_{-1}, [S, A]] \subseteq [[L_{-1}, S], A] + [S, [L_{-1}, A]] \subseteq [L_{-2}, A] + [S, L_0] \subseteq \{0\} + S = S$, and $[L_{-2}, [S, A]] = [S, [L_{-2}, A]] = \{0\}$, so by Lemma 5, $[S, A] = \{0\}$. Hence, $[S, [L_{-1}, A]] \subseteq [[S, L_{-1}], A] + [L_{-1}, [S, A]] \subseteq [L_{-2}, A] + \{0\} = \{0\}$, so $\{0\} = [L_{-1}, [L_{-2}, A]] = [L_{-2}, [L_{-1}, A]] \supseteq [[L_{-1}, S], [L_{-1}, A]] = [[L_{-1}, [L_{-1}, A]], S]$, so that $[L_{-1}, [L_{-1}, A]] \subseteq \mathrm{Ann}_{L_{-1}} S = S$. Since we also have $[L_{-2}, [L_{-1}, A]] = [L_{-1}, [L_{-2}, A]] = \{0\}$, it follows from Lemma 5 that $[L_{-1}, A] = \{0\}$, so that by Lemma 1, $A = \{0\}$. Q.E.D.

LEMMA 9. $\mathrm{Ann}_{L_i} L_{-2} = \{0\}$, $1 \leqslant i \leqslant k$.

PROOF. Since $[L_{-1}, \mathrm{Ann}_{L_i} L_{-2}] \subseteq \mathrm{Ann}_{L_{i-1}} L_{-2}$, $i > -1$, we prove this lemma by assuming the contrary and arriving at a contradiction using induction and Lemmas 1 and 8. Q.E.D.

We now assume that $L_{-1} = S + T$, where S and T are proper L_0-submodules of L_{-1}.

LEMMA 10. *$L_{-1} = S \oplus T$, where S and T are irreducible abelian L_0-submodules of L_{-1}.*

PROOF. By Lemma 4, S and T are abelian. If $v \in T \cap S$, then $[v, L_{-1}] = [v, T + S] \subseteq [v, T] + [v, S] = \{0\}$, since both T and S are abelian. Hence, $L_{-1} = T \oplus S$.

Let S_0 be a proper submodule of S. If we set $T_0 = T \oplus S_0$, then we have by Lemma 4 that T_0 is abelian. Then for any $s_0 \in S_0$, we have $[s_0, L_{-1}] = [s_0, T_0 + S] \subseteq [s_0, T_0] + [s_0, S] = \{0\}$, as before, so by Lemma 1, $s_0 = 0$. Hence, S can contain no proper submodule, so S is irreducible. Similarly, T is irreducible, also. Q.E.D.

LEMMA 11. $L_{-2} = [S, T]$.

PROOF. By Lemma 1, $L_{-2} = [L_{-1}, S] = [S \oplus T, S] \subseteq [S, S] + [T, S] = [T, S]$, since S is abelian. Q.E.D.

LEMMA 12. $\mathrm{Ann}_{L_0} L_{-2} \cap \mathrm{Ann}_{L_0} S = \{0\} = \mathrm{Ann}_{L_0} S \cap \mathrm{Ann}_{L_0} T$.

PROOF. Set $C = \mathrm{Ann}_{L_0} L_{-2} \cap \mathrm{Ann}_{L_0} S$. Then by definition, $[L_{-2}, C] = \{0\}$. Furthermore, $[L_{-1}, C] = [S \oplus T, C] \subseteq [S, C] + [T, C] \subseteq T$, which is abelian. Hence, by Lemma 5, $C = \{0\}$. Set $D = \mathrm{Ann}_{L_0} S \cap \mathrm{Ann}_{L_0} T$. Then by Lemma 11, we have $[L_{-2}, D] = [[S, T], D] = [[S, D], T] + [S, [D, T]] = \{0\}$, so $D \subseteq C = \{0\}$. Q.E.D.

LEMMA 13. $L_{2i-1} = [L_{2i}, S] \oplus [L_{2i}, T]$, $0 \leqslant i \leqslant [k/2]$.

PROOF. Since by Lemma 2, $L_{2i-1} = [L_{2i}, L_{-1}]$, we have by assumption that $L_{2i-1} \subseteq [L_{2i}, S] + [L_{2i}, T]$. If $v \in [L_{2i}, S] \cap [L_{2i}, T]$, we have $v(\mathrm{ad}\, L_{-2})^i \in S \cap T = \{0\}$, so by Lemma 9 and induction, $v = 0$. Q.E.D.

LEMMA 14. $\mathrm{Ann}_{L_{2i-1}} S = [L_{2i}, S]$, $0 \leqslant i \leqslant [k/2]$.

PROOF. Since $[[L_{2i}, S], S](\mathrm{ad}\, L_{-2})^i \subseteq [S, S] = \{0\}$, we have by Lemma 9 and induction that $[[L_{2i}, S], S] = \{0\}$. If $x \in (L_{2i-1} \setminus [L_{2i}, S]) \cap \mathrm{Ann}_{L_{2i-1}} S$, then by Lemma 13, $x = v + y$, where $0 \neq v \in [L_{2i}, T] \cap \mathrm{Ann}_{L_{2i-1}} S$, and $y \in [L_{2i}, S]$. Then $\{0\} = [v, S](\mathrm{ad}\, L_{-2})^i = [v(\mathrm{ad}\, L_{-2})^i, S]$, so $v(\mathrm{ad}\, L_{-2})^i \subseteq \mathrm{Ann}_{L_{-1}} S = S$ (see Lemma 10 and the remark following Lemma 7). But $v(\mathrm{ad}\, L_{-2})^i \subseteq [L_{2i}, T](\mathrm{ad}\, L_{-2})^i = [L_{2i}(\mathrm{ad}\, L_{-2})^i, T] \subseteq T$, so $v(\mathrm{ad}\, L_{-2})^i \subseteq S \cap T = \{0\}$. Thus, by Lemma 9 and induction, $v = 0$. Q.E.D.

LEMMA 15. $[[L_{2i}, S], [L_{2j}, S]] = \{0\}$, $0 \leqslant i, j \leqslant [k/2]$.

PROOF. $[[L_{2i}, S], [L_{2j}, S]](\mathrm{ad}\, L_{-2})^{i+j} \subseteq [S, S] = \{0\}$, so by Lemma 9 and induction, $[[L_{2i}, S], [L_{2j}, S]] = \{0\}$. Q.E.D.

LEMMA 16. $[L_{2i}, [L_{2j}, S]] \subseteq \mathrm{Ann}_{L_{2(i+j)-1}} S$.

PROOF. Since $[[L_{2i}, [L_{2j}, S]], S](\mathrm{ad}\, L_{-2})^{i+j} \subseteq [S, S] = \{0\}$, we have by Lemmas 9 and 14 that $[L_{2i}, [L_{2j}, S]] \subseteq \mathrm{Ann}_{L_{2(i+j)-1}} S$. Q.E.D.

LEMMA 17. *If k is odd, then L_k annihilates S or T. In the former case, we have $[L_{-2}, L_k] \subseteq [L_{k-1}, S]$. The latter case is similar.*

PROOF. By Lemmas 3 and 2, we have that $\{0\} \neq [L_0, L_k] = [[L_{-1}, L_1], L_k] = [L_{k-1}, L_1]$, so by Lemmas 2 and 13, $L_k = [L_{k-1}, L_1] = [L_{k-1}, [L_2, S] \oplus [L_2, T]] \subseteq [L_{k-1}, [L_2, S]] + [L_{k-1}, [L_2, T]]$. By Lemma 16, $[L_{k-1}, [L_2, S]] \subseteq \mathrm{Ann}_{L_k} S$, and since for any i, $-1 \leqslant i \leqslant k$, we have that $[L_{-2}, \mathrm{Ann}_{L_i} S] \subseteq \mathrm{Ann}_{L_{i-2}} S$, we must have by Lemma 14 that $[L_{k-1}, [L_2, S]](\mathrm{ad}\, L_{-2})^{((k+1)/2)} \subseteq \mathrm{Ann}_{L_{-1}} S \subseteq S$. Similarly, $[L_{k-1}, [L_2, T]](\mathrm{ad}\, L_{-2})^{((k+1)/2)} \subseteq T$, so if $v \in [L_{k-1}, [L_2, S]] \cap [L_{k-1}, [L_2, T]]$, then $v(\mathrm{ad}\, L_{-2})^{((k+1)/2)} \subseteq S \cap T = \{0\}$, so by Lemma 9 and induction, $v = 0$. Thus, $L_k = [L_{k-1}, [L_2, S]] \oplus [L_{k-1}, [L_2, T]]$. But by Lemma 2, L_k is an irreducible L_0-module, so either $[L_{k-1}, [L_2, S]] = \{0\}$ or $[L_{k-1}, [L_2, T]] = \{0\}$. In the latter case, we have $[L_{-2}, L_k] \subseteq \mathrm{Ann}_{L_{k-2}} S = [L_{k-1}, S]$, by Lemma 14. Q.E.D.

PROOF OF THEOREM. We will assume that $[L_k, S] = \{0\}$, if k is odd. The case where $[L_k, T] = \{0\}$ is symmetric. We set

$$M_{-1} = L_{-2} \oplus S,$$
$$M_0 = T \oplus L_0 \oplus [L_2, S],$$
$$M_i = [L_{2i}, T] \oplus L_{2i} \oplus [L_{2i+2}, S], \qquad 1 \leqslant i < [k/2],$$

and

$$M_{[k/2]} = \begin{cases} [L_k, T] \oplus L_k, & \text{if } k \text{ is even,} \\ [L_{k-1}, T] \oplus L_{k-1} \oplus L_k, & \text{if } k \text{ is odd.} \end{cases}$$

Then by Lemmas 10 and 13, $L = M = M_{-1} \oplus M_0 \oplus \cdots \oplus M_{[k/2]}$, and by Lemmas 6, 14, 15, 16, and 17, $[M_i, M_j] \subseteq M_{i+j}$, $-1 \leqslant i, j \leqslant [k/2]$. (If k is odd, then both $[L_2, [L_{k-1}, T]]$ and $[[L_2, T], L_{k-1}]$ annihilate T by Lemma 16. As subspaces of L_k, they annihilate S. Hence, they are zero by Lemma 1.)

It remains to show that M is irreducible and transitive. That M is irreducible follows from Lemma 1 of [2]. We must now show that M is transitive. By Lemma 9, we have that $\mathrm{Ann}_{M_i} M_{-1} = \{0\}$ for $i > 0$. Moreover, we have that

$$\mathrm{Ann}_{M_0} M_{-1} = (\mathrm{Ann}_T L_{-2} \cap \mathrm{Ann}_T S) \oplus (\mathrm{Ann}_{L_0} L_{-2} \cap \mathrm{Ann}_{L_0} S) \oplus (\mathrm{Ann}_{[L_2, S]} L_{-2} \cap \mathrm{Ann}_{[L_2, S]} S).$$

Because we have by Lemma 14 that $\mathrm{Ann}_T S \subseteq T \cap \mathrm{Ann}_{L_{-1}} S \subseteq T \cap S = \{0\}$, the first summand is zero. The second summand is zero by Lemma 12. The third summand is contained in $\mathrm{Ann}_{L_1} L_{-2}$, which is zero by Lemma 8. To complete the proof, we have to check that $\mathrm{Ann}_{M_{-1}} M_1 = \{0\}$ and $\mathrm{Ann}_{M_0} M_1 = \{0\}$, which we do in the following proof, which is due to the referee, to whom the present author is most gratefu.. That $\mathrm{Ann}_{M_{-1}} M_1 = \{0\}$ follows from the irreducibility of M_{-1} as an M_0-module and the fact that $[M_{-1}, M_1] \neq \{0\}$, which we have just demonstrated. Now let $M' = \sum M_i'$ be the subalgebra of M generated by $M_{-1} \oplus M_0 \oplus M_1$. Let I be the ideal of M' generated by $\mathrm{Ann}_{M_0} M_1$. Then by the Poincaré-Birkhoff-Witt Theorem,

$$\begin{aligned} I &= U(M')\mathrm{Ann}_{M_0} M_1 = U(M_{-1})U(M_0)U(M_1' \oplus M_2' \oplus \cdots)\mathrm{Ann}_{M_0} M_1 \\ &= U(M_{-1})\mathrm{Ann}_{M_0} M_1 \subseteq M_{-1} \oplus \mathrm{Ann}_{M_0} M_1. \end{aligned}$$

But $I \supseteq [I, M'] \supseteq [M_{-1}, M_1] \supseteq [L_{-2}, [L_2, T]] \supseteq T$ (by Lemmas 8 and 10 as $[L_{-2}, [L_2, T]] = [[L_{-2}, L_2], T]$). Thus, $[T, L_2] \subseteq I \cap M_1 = \{0\}$. Then by Lemmas 1 and 13, we would have $L_{-1} = [L_1, L_{-2}] = [[L_2, S] \oplus [L_2, T], L_{-2}] = [[L_2, S], L_{-2}] \subseteq [[L_2, L_{-2}], S] \subseteq S$, which, by Lemma 10, contradicts our assumption that L_{-1} is reducible. Q.E.D.

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Department of Mathematics, The Ohio State University at Mansfield, Mansfield, Ohio 44906