ON SIMPLE REDUCIBLE LIE ALGEBRAS OF DEPTH TWO

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ABSTRACT. We show that under certain circumstances simple finite-dimensional reducible graded Lie algebras of the form \( L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k \) can be given irreducible transitive gradations of the form \( M_{-1} \oplus M_0 \oplus \cdots \oplus M_{(k/2)} \).

In [1], the present author classified the simple finite-dimensional irreducible graded Lie algebras over an algebraically closed field of characteristic \( p > 5 \) which have the form

\[
L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_k \tag{1}
\]

where \( L_0 \) is classical and reductive and \( k > 3 \). In this paper, we consider certain reducible Lie algebras \( L \) of the form (1). In them, we show how to construct an irreducible, transitive gradation

\[
L = M_{-1} \oplus M_0 \oplus \cdots \oplus M_{(k/2)} \tag{2}
\]

Theorem. Let \( L \) be a simple finite-dimensional reducible graded Lie algebra of the form (1) over a field of any characteristic, and suppose that \( L_0 \) contains no nonzero abelian ideal which annihilates \( L_{-2} \) and that \( L_{-1} = S + T \), where \( S \) and \( T \) are proper \( L_0 \)-submodules of \( L_{-1} \). Then \( S \) and \( T \) are irreducible abelian \( L_0 \)-submodules, and \( L_{-1} = S \oplus T \). In addition, \( L \) possesses an irreducible transitive gradation of the form (2).

We will prove this theorem by means of a series of lemmas. We begin by restating the first three lemmas of [1]. Their proofs there are valid in this setting, also.

**Lemma 1.** If \( L \) is a simple graded Lie algebra of the form (1), then \( L_{-2} \) is an irreducible \( L_0 \)-module. Furthermore, \([L_{-1}, L_{-1}] \neq \{0\}\), so \([L_{-1}, L_{-1}] = L_{-2}\); also, \( L_{-1} = [L_{-2}, L_1] \). Lastly, \([L_{-1}, x] \neq \{0\} \) for all \( x \in L_{-2} \).

**Lemma 2.** If \( L \) is a simple graded Lie algebra of the form (1), then \( L_k \) is an irreducible \( L_0 \)-module, and \( L_{j-1} = [L_j, L_{-1}] \), for \(-1 < j < k\).

**Lemma 3.** We have \([L_{-2}, L_i] \neq \{0\} \), \( 0 < i < k \), and \([L_0, L_k] \neq \{0\} \).

**Lemma 4.** \( L_{-1} \) contains no proper nonabelian \( L_0 \)-submodule.

**Proof.** Let \( N \) be a nonzero nonabelian \( L_0 \)-submodule of \( L_{-1} \). Then \([N, N] \) is a nonzero \( L_0 \)-submodule of the irreducible \( L_0 \)-submodule \( L_{-2} \), so \([N, N] = L_{-2} \).

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Then we have by Lemma 1 that $L_{-1} = [L_{-2}, L_1] = [[N, N], L_1] \subseteq [[N, L_1], N] + [N, [N, L_1]] \subseteq [L_0, N] + [N, L_0] \subseteq N$, so that $N = L_{-1}$. Q.E.D.

Lemmas 5 through 9 are valid for any simple finite-dimensional $L$ of the form (1) where $L_0$ contains no nonzero abelian ideal which annihilates $L_{-2}$ and where $L_{-1}$ contains a proper $L_0$-submodule $S$.

**Lemma 5.** Let $X$ be an $L_0$-submodule of $L_0$ such that $[L_{-1}, X]$ is abelian and $[L_{-2}, X] = \{0\}$. Then $X = \{0\}$.

**Proof.** We have $[L_{-1}, [[L_{-1}, X], X]] \subseteq [L_{-1}, [L_{-1}, X], X] + [[L_{-1}, X], [L_{-1}, X]] \subseteq [L_{-2}, X] + \{0\} = \{0\}$. Thus, by Lemma 1, $[[L_{-1}, X], X] = \{0\}$. Then $[L_{-1}, [X, X]] \subseteq [L_{-1}, X, X] + [X, [L_{-1}, X]] = \{0\}$, so $[X, X] = \{0\}$. Since $L_0$ is assumed to contain no nonzero abelian ideal which annihilates $L_{-2}$, we must have $X = \{0\}$. Q.E.D.

**Lemma 6.** $[S, [S, L_2]] = \{0\}$.

**Proof.** We have that $[L_{-1}, [S, [S, L_2]]] \subseteq [L_{-1}, S], [S, L_2]] + [S, [[L_{-1}, S], L_2]] + [S, [S, [L_{-1}, L_2]]] \subseteq [S, [[L_{-1}, S], L_2]] + [S, L_0] + [S, L_0] \subseteq S$, which is abelian by Lemma 4. Furthermore, $[L_{-2}, [S, [S, L_2]]] = [S, [S, [L_{-2}, L_2]]] \subseteq [S, [S, L_0]] \subseteq [S, S] = \{0\}$. Thus, by Lemma 5, $[S, [S, L_2]] = \{0\}$. Q.E.D.

**Lemma 7.** $S$ is contained in a maximal $L_0$-submodule $U$ of $L_{-1}$ such that $U = \text{Ann}_{L_{-1}} U$.

**Proof (Referee).** Let $U$ be a maximal proper $L_0$-submodule of $L_{-1}$ containing $S$. Then by Lemma 4, $\{0\} = [U, U] = [U, S]$, so $U \subseteq \text{Ann}_{L_{-1}} U \subseteq \text{Ann}_{L_{-1}} S$. Since $[L_{-1}, S] \neq \{0\}$ (by Lemma 1), the maximality of $U$ forces $\hat{U} = \text{Ann}_{L_{-1}} \hat{U} = \text{Ann}_{L_{-1}} S$. Q.E.D.

In what follows, we let $S = U$; i.e., we assume that $S$ is a maximal $L_0$-submodule of $L_{-1}$ which is equal to its own annihilator.

**Lemma 8.** $\text{Ann}_{L_{-1}} L_{-2} = \{0\}$.

**Proof.** Set $A = \text{Ann}_{L_{-1}} L_{-2}$. Then $[L_{-1}, [S, A]] \subseteq [[L_{-1}, S], A] + [S, [L_{-1}, A]] \subseteq [L_{-2}, A] + [S, L_0] \subseteq \{0\} + S = S$, and $[L_{-2}, [S, A]] = [S, [L_{-2}, A]] = \{0\}$, so by Lemma 5, $[S, A] = \{0\}$. Hence, $[S, [L_{-1}, A]] \subseteq [[S, L_{-1}], A] + [L_{-1}, [S, A]] \subseteq [L_{-2}, A] + \{0\} = \{0\}$, so $[L_{-1}, [L_{-2}, A]] = [L_{-2}, [L_{-1}, A]] \supseteq [[L_{-1}, S], [L_{-1}, A]] = [[L_{-1}, L_{-1}], S]$, so that $[L_{-1}, [L_{-1}, A]] \subseteq \text{Ann}_{L_{-1}} S = S$. Since we also have $[L_{-2}, [L_{-1}, A]] = [L_{-1}, [L_{-2}, A]] = \{0\}$, it follows from Lemma 5 that $[L_{-1}, A] = \{0\}$, so that by Lemma 1, $A = \{0\}$. Q.E.D.

**Lemma 9.** $\text{Ann}_{L_{-i}} L_{-2} = \{0\}$, $1 < i < k$.

**Proof.** Since $[L_{-1}, \text{Ann}_{L_{-i}} L_{-2}] \subseteq \text{Ann}_{L_{-i-1}} L_{-2}$, $i > -1$, we prove this lemma by assuming the contrary and arriving at a contradiction using induction and Lemmas 1 and 8. Q.E.D.

We now assume that $L_{-1} = S + T$, where $S$ and $T$ are proper $L_0$-submodules of $L_{-1}$.
Lemma 10. \( L_{-1} = S \oplus T \), where \( S \) and \( T \) are irreducible abelian \( L_0 \)-submodules of \( L_{-1} \).

Proof. By Lemma 4, \( S \) and \( T \) are abelian. If \( v \in T \cap S \), then \([v, L_{-1}] = [v, T + S] \subseteq [v, T] + [v, S] = \{0\}\), since both \( T \) and \( S \) are abelian. Hence, \( L_{-1} = T \oplus S \).

Let \( S_0 \) be a proper submodule of \( S \). If we set \( T_0 = T \oplus S_0 \), then we have by Lemma 4 that \( T_0 \) is abelian. Then for any \( s_0 \in S_0 \), we have \([s_0, L_{-1}] = [s_0, T_0 + S] \subseteq [s_0, T_0] + [s_0, S] = \{0\}\), as before, so by Lemma 1, \( s_0 = 0 \). Hence, \( S \) cannot contain any proper submodule, so \( S \) is irreducible. Similarly, \( T \) is irreducible, also. Q.E.D.

Lemma 11. \( L_{-2} = [S, T] \).

Proof. By Lemma 1, \( L_{-2} = [L_{-1}, S] = [S \oplus T, S] \subseteq [S, S] + [T, S] = [T, S] \), since \( S \) is abelian. Q.E.D.

Lemma 12. \( \text{Ann}_L L_{-2} \cap \text{Ann}_L S = \{0\} = \text{Ann}_L S \cap \text{Ann}_L T \).

Proof. Set \( C = \text{Ann}_L L_{-2} \cap \text{Ann}_L S \). Then by definition, \([L_{-2}, C] = \{0\}\). Furthermore, \([L_{-1}, C] = [S \oplus T, C] \subseteq [S, C] + [T, C] \subseteq T \), which is abelian. Hence, by Lemma 5, \( C = \{0\} \). Set \( D = \text{Ann}_L S \cap \text{Ann}_L T \). Then by Lemma 11, we have \([L_{-2}, D] = [[S, T], D] = [[S, D], T] + [S, [D, T]] = \{0\}\), so \( D \subseteq C = \{0\} \). Q.E.D.

Lemma 13. \( L_{2i-1} = [L_{2i}, S] \oplus [L_{2i}, T], 0 < i < [k/2] \).

Proof. Since by Lemma 2, \( L_{2i-1} = [L_{2i}, L_{-1}] \), we have by assumption that \( L_{2i-1} \subseteq [L_{2i}, S] + [L_{2i}, T] \). If \( v \in [L_{2i}, S] \cap [L_{2i}, T] \), we have \( v(\text{ad} L_{-2}) \in S \cap T = \{0\} \), so by Lemma 9 and induction, \( v = 0 \). Q.E.D.

Lemma 14. \( \text{Ann}_L L_{2i-1, S} = [L_{2i}, S], 0 < i < [k/2] \).

Proof. Since \([L_{2i-1}, S], S(\text{ad} L_{-2}) \subseteq [S, S] = \{0\}\), we have by Lemma 9 and induction that \([L_{2i-1}, S], S] = \{0\}\). If \( x \in ([L_{2i-1} \setminus [L_{2i}, S]] \cap \text{Ann}_L L_{2i-1, S} \), then by Lemma 13, \( x = v + y \), where \( 0 \neq v \in [L_{2i}, T] \cap \text{Ann}_L L_{2i-1, S} \), and \( y \in [L_{2i}, S] \). Then \( \{0\} = [v, S(\text{ad} L_{-2})] = [v(\text{ad} L_{-2}), S], \) so \( v(\text{ad} L_{-2}) \subseteq \text{Ann}_L L_{2i-1, S} = S \) (see Lemma 10 and the remark following Lemma 7). But \( v(\text{ad} L_{-2}) \subseteq [L_{2i}, T] = [L_{2i}, L_{2i-1} \cap T = \{0\}] \). Thus, by Lemma 9 and induction, \( v = 0 \). Q.E.D.

Lemma 15. \([L_{2i}, S], [L_{2j}, S]] = \{0\}, 0 < i, j < [k/2] \).

Proof. \([L_{2i}, S], [L_{2j}, S]](\text{ad} L_{-2})^{i+j} \subseteq [S, S] = \{0\}\), so by Lemma 9 and induction, \([L_{2i}, S], [L_{2j}, S]] = \{0\}\). Q.E.D.

Lemma 16. \([L_{2i}, [L_{2j}, S]] \subseteq \text{Ann}_L L_{2i+n, S} \).

Proof. Since \([L_{2i}, [L_{2j}, S]], S(\text{ad} L_{-2})^{i+j} \subseteq [S, S] = \{0\}\), we have by Lemmas 9 and 14 that \([L_{2i}, [L_{2j}, S]] \subseteq \text{Ann}_L L_{2i+n, S} \). Q.E.D.

Lemma 17. If \( k \) is odd, then \( L_k \) annihilates \( S \) or \( T \). In the former case, we have \([L_{-2}, L_k] \subseteq [L_{k-1}, S] \). The latter case is similar.
Proof. By Lemmas 3 and 2, we have that $\{0\} \neq [L_0, L_k] = [L_{-1}, L_1], L_k = [L_{k-1}, L_1] = [L_0, S] \oplus [L_2, T] \subseteq [L_{k-1}, [L_2, S]] + [L_{k-1}, [L_2, T]].$ By Lemma 16, $[L_{k-1}, [L_2, S]] \subseteq \text{Ann}_{L_k} S$, and since for any $i, -1 < i < k$, we have that $[L_{-1}, \text{Ann}_{L_i} S \subseteq \text{Ann}_{L_{i+1}} S$, we must have by Lemma 14 that $[L_{k-1}, [L_2, S]](ad L_{k-1})^{(2)} \subseteq \text{Ann}_{L_{k-1}} S \subseteq S$. Similarly, $[L_{k-1}, [L_2, T]](ad L_{k-1})^{(2)} \subseteq T$, so if $v \in [L_{k-1}, [L_2, S]] \cap [L_{k-1}, [L_2, T]]$, then $v(ad L_{k-1})^{(2)} \subseteq S \cap T = \{0\}$, so by Lemma 9 and induction, $v = 0$. Thus, $L_k = [L_{k-1}, [L_2, S]] \oplus [L_{k-1}, [L_2, T]]$. But by Lemma 2, $L_k$ is an irreducible $L_0$-module, so either $[L_{k-1}, [L_2, S]] = \{0\}$ or $[L_{k-1}, [L_2, T]] = \{0\}$. In the latter case, we have $[L_{-2}, L_k] \subseteq \text{Ann}_{L_{k-2}} S = [L_{k-1}, S]$, by Lemma 14.

Q.E.D.

Proof of Theorem. We will assume that $[L_k, S] = \{0\}$, if $k$ is odd. The case where $[L_k, T] = \{0\}$ is symmetric. We set

$$M_0 = [L_2, T] \oplus L_1 \oplus [L_2, S],$$

and

$$M_i = [L_{2i}, T] \oplus L_{2i} \oplus [L_{2i+2}, S], \quad 1 < i < \lfloor k/2 \rfloor,$$

and

$$M_{\lfloor k/2 \rfloor} = \begin{cases} [L_k, T] \oplus L_k, & \text{if } k \text{ is even}, \\ [L_{k-1}, T] \oplus L_{k-1} \oplus L_k, & \text{if } k \text{ is odd}. \end{cases}$$

Then by Lemmas 10 and 13, $L = M = M_1 \oplus M_0 \oplus \cdots \oplus M_{\lfloor k/2 \rfloor}$, and by Lemmas 6, 14, 15, 16, and 17, $[M_i, M_j] \subseteq M_{i+j}, -1 < i, j < \lfloor k/2 \rfloor$. (If $k$ is odd, then both $[L_2, [L_{k-1}, T]]$ and $[[L_2, T], L_{k-1}]$ annihilate $T$ by Lemma 16. As subspaces of $L_k$, they annihilate $S$. Hence, they are zero by Lemma 1.)

It remains to show that $M$ is irreducible and transitive. That $M$ is irreducible follows from Lemma 1 of [2]. We must now show that $M$ is transitive. By Lemma 9, we have that $\text{Ann}_{M_i} M_{i-1} = \{0\}$ for $i > 0$. Moreover, we have that

$$\text{Ann}_{M_0} M_{i-1} = (\text{Ann}_T L_{-2} \cap \text{Ann}_T S) \oplus (\text{Ann}_{L_0} L_{-2} \cap \text{Ann}_{L_0} S) \oplus (\text{Ann}_{[L_2, S]} L_{-2} \cap \text{Ann}_{[L_2, S]} S).$$

Because we have by Lemma 14 that $\text{Ann}_T S \subseteq T \cap \text{Ann}_{L_{-1}} S \subseteq T \cap S = \{0\}$, the first summand is zero. The second summand is zero by Lemma 12. The third summand is contained in $\text{Ann}_{L_1} L_{-2}$, which is zero by Lemma 8. To complete the proof, we have to check that $\text{Ann}_{M_1} M_1 = \{0\}$ and $\text{Ann}_{M_0} M_1 = \{0\}$, which we do in the following proof, which is due to the referee, to whom the present author is most grateful. That $\text{Ann}_{M_1} M_1 = \{0\}$ follows from the irreducibility of $M_1$ as an $M_0$-module and the fact that $[M_{-1}, M_1] \neq \{0\}$, which we have just demonstrated.

Now let $M' = \Sigma M_i$ be the subalgebra of $M$ generated by $M_{-1} \oplus M_0 \oplus M_1$. Let $I$ be the ideal of $M'$ generated by $\text{Ann}_{M_0} M_1$. Then by the Poincaré-Birkhoff-Witt Theorem,

$$I = U(M')\text{Ann}_{M_0} M_1 = U(M_1)U(M_0)U(M_1' \oplus M_2' \oplus \cdots \text{Ann}_{M_0} M_1$$

$$= U(M_{-1})\text{Ann}_{M_0} M_1 \subseteq M_{-1} \oplus \text{Ann}_{M_0} M_1.$$

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But \( I \supseteq [I, M'] \supseteq [M_{-1}, M_1] \supseteq [L_{-2}, [L_2, T]] \supseteq T \) (by Lemmas 8 and 10 as \([L_{-2}, [L_2, T]] = [[L_{-2}, L_2], T]\)). Thus, \([T, L_2] \subseteq I \cap M_1 = \{0\}\). Then by Lemmas 1 and 13, we would have \( L_{-1} = [L_1, L_{-2}] = [[L_2, S] \oplus [L_2, T], L_{-2}] = [[L_2, S], L_{-2}] \subseteq [[L_2, L_{-2}], S] \subseteq S \), which, by Lemma 10, contradicts our assumption that \( L_{-1} \) is reducible. Q.E.D.

**References**


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