SUBADDITIVITY OF HOMOGENEOUS NORMS
ON CERTAIN NILPOTENT LIE GROUPS

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Abstract. Let $N$ be a Lie group with its Lie algebra generated by the left-invariant vector fields $X_1, \ldots, X_k$ on $N$. An explicit fundamental solution for the (hypoelliptic) operator $L = X_1^2 + \cdots + X_k^2$ on $N$ has been obtained for the Heisenberg group by Folland [1] and for the nilpotent (Iwasawa) groups of isometries of rank-one symmetric spaces by Kaplan and Putz [2]. Recently Kaplan [3] introduced a (still larger) class of step-2 nilpotent groups $N$ arising from Clifford modules for which similar explicit solutions exist. As in the case of $L$ being the ordinary Laplacian on $N = \mathbb{R}^n$, these solutions are of the form $g \mapsto \text{const} \|g\|^{2-m}$, $g \in N$, where the "norm" function $\| \|$ satisfies a certain homogeneity condition.

We prove that the above norm is also subadditive.

Let $u$, $v$ be real finite-dimensional vector spaces each equipped with a positive definite quadratic form $| \cdot |^2$. Let $\mu: u \times v \to v$ be a composition of these quadratic forms [3, p. 148] normalized in the sense that $\mu(u_0, v) = v$ for some $u_0 \in u$. Define $\phi: v \times v \to u$ by demanding $\langle u, \phi(v, v') \rangle = \langle \mu(u, v), v' \rangle, u \in u; v, v' \in v$, relative to the inner products $\langle , \rangle$ induced by the given quadratic forms. Let $\delta$ denote the orthogonal complement to $\mathbb{R}u_0$ in $u$ and $\pi: u \to \delta$ the orthogonal projection. Now set $n = v \times \delta$ and define a bracket on $n$ by $[[v, z], (v', z')] = (0, \pi \circ \phi(v, v'))$. On the simply connected analytic group $N$, corresponding to the Lie algebra $n$ (i.e. on Kaplan's type $H$ group) we define a norm function by $\|n\| = (|v|^4 + 16|z|^2)^{1/4}$, where $n = \exp(v + z), v \in v, z \in \delta; n \approx v \oplus \delta$. We now prove

Theorem. The norm function $\| \|$ is subadditive, i.e.

$$\|nn'\| \leq \|n\| + \|n'\|, \quad n, n' \in N.$$

Proof. We have

$$\|nn'\|^4 = \|\exp(v + v' + z + z' + \tfrac{1}{2}[v, v'])\|^4$$

$$= \|v + v'|^4 + 16|z + z'|^2 + \tfrac{1}{2}[v, v']|^2.$$

Now

$$|v + v'|^4 = |v|^4 + |v'|^4 + 4\langle v, v' \rangle^2 + 4|v|^2\langle v, v' \rangle$$

$$+ 4|v'|^2\langle v, v' \rangle + 2|v|^2|v'|^2,$$ (1)
$16|z + z' + \frac{1}{2}[v, v']|^2 = 16|z|^2 + 16|z'|^2 + 4|v, v'||^2$

$+ 16\langle z, [v, v'] \rangle + 16\langle z', [v, v'] \rangle + 32\langle z, z' \rangle. \quad (2)$

Since $2|v|^2|v'|^2 + 32\langle z, z' \rangle < 2\|n\|^2\|n'\|^2$ and

$4|v|^2\langle v, v' \rangle + 16\langle z, [v, v'] \rangle < 4\|n\|^2\left(\langle v, v' \rangle^2 + |[v, v']|^2\right)^{1/2},$

we need

**Lemma.** In the notation above

$\langle v, v' \rangle^2 + |[v, v']|^2 < |v|^2|v'|^2,$ \quad $v, v' \in b.$

For we have $|v| |v'| < \|n\| \|n'\|,$ and collecting the above inequalities we obtain

$(1) + (2) < (\|n\| + \|n'\|)^4.$

Proof of the Lemma follows from Schwarz's inequality on the hermitian form $h_z$

on $b$ defined by

$h_z(v, v') = \langle v, v' \rangle - \sqrt{-1} \langle z, \pi \circ \phi(v, v') \rangle,$

if one regards $b$ as a complex vector space under the complex structure $J_z: b \rightarrow b$

given by $\langle J_z(v), v' \rangle = \langle z, \phi(v, v') \rangle$ with fixed $z \in \mathfrak{g}, \ |z| = 1$ (see [3, pp. 149, 150]),

simply by putting $z = [v, v']/|[v, v']|.$

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**References**


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