SUBADDITIVITY OF HOMOGENEOUS NORMS
ON CERTAIN NILPOTENT LIE GROUPS

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Abstract. Let $N$ be a Lie group with its Lie algebra generated by the left-invariant vector fields $X_1, \ldots, X_k$ on $N$. An explicit fundamental solution for the (hypoelliptic) operator $L = X_1^2 + \cdots + X_k^2$ on $N$ has been obtained for the Heisenberg group by Folland [1] and for the nilpotent (Iwasawa) groups of isometries of rank-one symmetric spaces by Kaplan and Putz [2]. Recently Kaplan [3] introduced a (still larger) class of step-2 nilpotent groups $N$ arising from Clifford modules for which similar explicit solutions exist. As in the case of $L$ being the ordinary Laplacian on $N = \mathbb{R}^n$, these solutions are of the form $g \mapsto \text{const} \|g\|^{2-m}$, $g \in N$, where the “norm” function $\| \|$ satisfies a certain homogeneity condition. We prove that the above norm is also subadditive.

Let $u, v$ be real finite-dimensional vector spaces each equipped with a positive definite quadratic form $| \cdot |^2$. Let $\mu : u \times v \to v$ be a composition of these quadratic forms [3, p. 148] normalized in the sense that $\mu(u_0, v) = v$ for some $u_0 \in u$. Define $\phi : v \to u$ by demanding $\langle u, \phi(v, v') \rangle = \langle \mu(u, v), v' \rangle$, $u \in u; v, v' \in v$, relative to the inner products $\langle \cdot, \cdot \rangle$ induced by the given quadratic forms. Let $\delta$ denote the orthogonal complement to $\text{R}u_0$ in $u$ and $\pi : u \to \delta$ the orthogonal projection. Now set $n = v \times \delta$ and define a bracket on $n$ by $[(v, z), (v', z')] = (0, \pi \circ \phi(v, v'))$. On the simply connected analytic group $N$, corresponding to the Lie algebra $n$ (i.e. on Kaplan’s type H group) we define a norm function by $\|n\| = (|v|^4 + 16|z|^2)^{1/4}$, where $n = \exp(v + z), v \in v, z \in \delta; n \cong v \oplus \delta$. We now prove

Theorem. The norm function $\| \|$ is subadditive, i.e.

$$\|nn'\| \leq \|n\| + \|n'\|, \quad n, n' \in N.$$  

Proof. We have

$$\|nn'\|^4 = \|\exp(v + v' + z + z' + \frac{1}{2}[v, v'])\|^4$$

$$= \|v + v'|^4 + 16|z + z' + \frac{1}{2}[v, v']|^2.$$  

Now

$$|v + v'|^4 = |v|^4 + |v'|^4 + 4\langle v, v' \rangle^2 + 4|v|^2\langle v, v' \rangle$$

$$+ 4|v'|^2\langle v, v' \rangle + 2|v|^2|v'|^2,$$

(1)
\[16|z + z' + \frac{1}{2} [v, v']|^2 = 16|z|^2 + 16|z'|^2 + 4[v, v']^2 + 16\langle z, [v, v'] \rangle + 16\langle z', [v, v'] \rangle + 32\langle z, z' \rangle.\]  \hspace{1cm} (2)

Since \(2|v|^2|v'|^2 + 32\langle z, z' \rangle < 2||n||^2||n'||^2\) and
\[4|v|^2\langle v, v' \rangle + 16\langle z, [v, v'] \rangle < 4||n||^2(\langle v, v' \rangle^2 + [v, v']^2)^{1/2},\]
we need

**Lemma.** In the notation above
\[\langle v, v' \rangle^2 + ||[v, v']||^2 < ||v||^2|v'|^2, \quad v, v' \in \mathcal{V}.

For we have \(|v| |v'| < ||n|| ||n'||\), and collecting the above inequalities we obtain
\[(1) + (2) < (||n|| + ||n'||)^4.

Proof of the Lemma follows from Schwarz’s inequality on the hermitian form \(h_z\) on \(\mathcal{V}\) defined by
\[h_z(v, v') = \langle v, v' \rangle - \sqrt{-1} \langle z, \pi \circ \phi(v, v') \rangle,
\]
if one regards \(\mathcal{V}\) as a complex vector space under the complex structure \(J_z: \mathcal{V} \rightarrow \mathcal{V}\) given by \(\langle J_z(v), v' \rangle = \langle z, \phi(v, v') \rangle\) with fixed \(z \in \mathcal{V}, |z| = 1\) (see [3, pp. 149, 150]), simply by putting \(z = [v, v']/[|v, v'|]\).

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**References**


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