**Abstract.** By observing a convex property of discrete differences, one-sided $k$-discrete, $k$-discrete Gâteaux and $k$-discrete Fréchet differentials are introduced. It is proved that a locally bounded $n$-convex function has $k$-discrete Fréchet differentials for $1 < k < n - 2$ and one-sided $(n - 1)$-discrete differentials at every point of its domain. Various properties of discrete differentials of an $n$-convex function are studied. As an application of these results the author proves that an $n$-convex function has a strong $(n - 2)$-Taylor series expansion and an $(n - 1)$th Fréchet differential provided it has a strong $n$-Taylor series expansion about the point.

1. Introduction. Higher differentials of a function on a Banach space can be defined either inductively with reference to the lower differentials or directly with no reference to lower order differentials. Riemann (see Butzer and Kozakiewez [2]), Peano $(H,\beta)$-Peano (see Weil [13], Ash [1]), and Taylor differentials (see Nashed [11]) are introduced in literature to define higher differentiability directly without reference to lower ones. One-sided $k$-discrete, $k$-discrete Gâteaux and $k$-discrete Fréchet differentials are introduced as direct differentials by using the discrete difference notion (see Dayal [5]), which is the extension to vector-valued functions of the divided differences of numerical analysis. The notion also induces a class of functions called $n$-convex functions, given by the author in 1972 in [4], which was introduced in literature by various approaches.

The class of $n$-convex functions contains the class of subconvex functions commonly known as convex functions ($n = 2$), the monotonic functions ($n = 1$), and the class of positive-valued functions. The first main result (Theorem 3.2) states that the $k$-discrete differential for $1 < k < n - 2$ and one-sided $(n - 1)$-discrete differential of a locally bounded $n$-convex function exist at every point of its domain. The second main result (Theorem 3.4) states that the one-sided $k$-discrete differential of an $n$-convex function satisfies a kind of uniform continuity (see Definition 3.3) for $1 < k < n - 2$. These results together with Theorem 3.5 and a local representation theorem (see Theorem 3.2 in [5]) are used by the author to prove that $n$-convex functions which are locally bounded admit a strong $(n - 2)$ Taylor series expansion (for definition, see [5]), which, in turn, proves the existence of the $(n - 1)$th Fréchet differential of the function. This result was proved by the author in 1972 [4] and the detailed proof will be given elsewhere.
2. Preliminaries. The meaning of differentiability and higher differentiability in the inductive sense is elegantly presented by Dieudonné [7] (see also [5]). The notions of one-sided Gâteaux, Gâteaux, and Fréchet differentiabilities along with the various direct differentials can be found in Nashed [12] (see also [4], [5]). We give the notion of discrete differences, which introduces the concept of one-sided \( k \)-discrete, \( k \)-discrete Gâteaux and \( k \)-discrete Fréchet differentials.

An \( n \)-discrete difference \([\Delta_n h](t_0, \ldots, t_n)\) of a function \( h: [a, b] \to F\), where \((t_0, \ldots, t_n)\) is a finite sequence of distinct numbers in the interval \([a, b]\) is the coefficient of \( t^n \) in the unique polynomial \( P(t) \) of degree \( < n \) such that \( P(t_i) = h(t_i) \) for \( i = 0, 1, 2, \ldots, n \). Thus

\[
[\Delta_n h](t_0, \ldots, t_n) = \sum_{k=0}^{n} \frac{1}{\prod_{j \neq k} (t_k - t_j)} \cdot h(t_k) \tag{2.1}
\]

and it is shown in [5] that the discrete difference satisfies a kind of convex property, namely, for \( t_0 < t < t_n \) with \( t_i \neq t_j \) for \( i \neq j \) and \( t \neq t_j, j = 1, 2, \ldots, n \),

\[
[\Delta_n h](t_0, \ldots, t_n) = \frac{(t - t_0)}{(t_n - t_0)} \cdot [\Delta_n h](t_0, \ldots, t_{n-1}, t) + \left( 1 - \frac{(t - t_0)}{(t_n - t_0)} \right) [\Delta_n h](t_1, \ldots, t_n, t).
\]

Let \( E \) and \( F \) be two Banach spaces and \( A \) be an open subset of \( E \). Let \( f: A \to F \). For every \( y \in A \) and \( v \in E \) we define a function \([h(y, v)]_t = f(y + tv)\). Let \([\Delta_k h(y, v)](t_0, \ldots, t_k)\) be the \( k \)-discrete difference of \( h(y, v) \) defined for any finite sequence \( t = (t_0, \ldots, t_k) \) with \( t_i \neq t_j \), \( i \neq j \), and such that \( y + tv \) is in the domain of \( f \) for all \( j = 0, 1, \ldots, k, y \in A \) and \( v \in E \). We write

\[
[\Delta_k h(y, v)](t_0, \ldots, t_k) = [\Delta_k h(y, v)]_t.
\]

**Definition 2.1.** Let \( E \) and \( F \) be Banach spaces and \( A \) be an open set of \( E \). For a function \( f: A \to F \) and some \( y \in A \) suppose that the limit

\[
(k!) \lim_{t \to 0+} [\Delta_k h(y, v)]_t \tag{2.2}
\]

exists for all \( v \in E \) in the sense that, for a given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( t = (t_0, \ldots, t_k) \) with \( 0 < t_i < \delta; t_i \neq t_j, i \neq j, i, j = 0, 1, \ldots, k, \)

\[
\|[k!] [\Delta_k h(y, v)]_t - (\hat{f}^{(k)} y)v|| < \varepsilon \quad \text{ (2.3)}
\]

Then \((\hat{f}^{(k)} y)\) is called the one-sided \( k \)-discrete differential of \( f \) at \( y \). A function \( f: A \to F \) is said to have a \( k \)-discrete Gâteaux differential \((f^{(k)} y): E^k \to F\) if \((f^{(k)} y)\) is a \( k \)-linear operator such that

\[
(f^{(k)} y)u = (f^{(k)} y)(u, u, \ldots, u),
\]

provided the limit (2.2) exists.

In addition, if the limit (2.2) is uniform for vectors \( v \) such that \( \|v\| < 1 \), then \((f^{(k)} y)\) is called the \( k \)-discrete Fréchet differential. 1-discrete Gâteaux and 1-discrete bounded Fréchet differentials are equivalent to Gâteaux and Fréchet differentials,
respectively. Further, the existence of the $k$-discrete Gâteaux or $k$-discrete Fréchet differential does not imply the existence of the corresponding $(k - 1)$-discrete differential.

3. Discrete differentials of an $n$-convex function. Discrete differences are used to define a class of functions called $n$-convex functions. R. Ger ([9], [10]) has defined these by using the finite distinct sequence with equal spacing, whereas we use the arbitrary finite distinct sequence. It is proved that the $k$-discrete Fréchet differential for $1 < k < n - 2$ and the one-sided $(n - 1)$-discrete differential of a locally bounded $n$-convex function exist at every point of its domain.

Definition 3.1. Let $A$ be an open subset of a Banach space $E$. The $k$-discrete difference $\Delta_k h(y, v)$ associated with $f: A \to \mathbb{R}$ is said to be monotonically increasing if for every $y \in A$ and $v \in E$,

$$[[\Delta_k h(y, v)]_t < [[\Delta_k h(y, v)]_{t^*}$$

for any sequences $t, t^*$ such that $y + t_i v$ and $y + t^*_i v$ are in $A$ for $i = 0, 1, \ldots, k$, and $t < t^*$ in the sense that $t_i < t^*_i$ for $i = 0, 1, \ldots, k$.

Definition 3.2. Let $A$ be an open subset of a Banach space $E$. The $k$-discrete difference $\Delta_k h(y, v)$ associated with $f: A \to \mathbb{R}$ is said to be uniformly bounded in the $\delta$-neighbourhood $N_\delta$ of $y_0 \in A$ if there exists a $\delta' > 0$ and a constant $M$ such that whenever $|t_i| < \delta'$, $y \in N_\delta$ and $||v|| < 1$, then $y + t_i v \in A$ and

$$|[[\Delta_k h(y, v)]_t| < M.$$ 

Definition 3.3. Let $A$ be an open subset of a Banach space $E$. The $k$-discrete difference $\Delta_k h(y, v)$ associated with $f: A \to \mathbb{R}$ is said to be strongly uniformly continuous in the $\delta/2$-neighbourhood $N_{\delta/2}$ of $y_0 \in A$ if, for a given $\epsilon > 0$, there is a $\delta'' > 0$ such that whenever $y \in N_{\delta/2}$, $|t_i| < \delta'/4$, $||v|| < 1$, $|t_i| < \delta'/4$ and $\sup_{0 < i < n} |t_i - s_i| < \delta''$ then

$$|[[\Delta_k h(y, v)]_t - [[\Delta_k h(y, v)]_{s}| < \epsilon.$$ 

Definition 3.4. Let $A$ be an open set of a Banach space $E$. A function $f: A \to \mathbb{R}$ is said to be $n$-convex if for all $y \in A$ and $v \in E$, the function $\Delta_n h(y, v)$ does not change sign as a function. $f t = (t_0, \ldots, t_n)$.

Theorem 3.1. Let $A$ be an open subset of a Banach space $E$ and $f: A \to \mathbb{R}$.

(a) If $f$ is $n$-convex then $\Delta_{n-1} h(y, v)$ is monotonic.

(b) If $f$ is $n$-convex and bounded in a neighborhood $N_\delta$ of $y_0$ then:

(i) $\Delta_{n-1} h(y, v)$ is uniformly bounded in $N_{\delta/2}$,

(ii) $\Delta_{n-1} h(y, v)$ is strongly uniformly continuous in $N_{\delta/2}$, $0 < k < n - 2$.

Proof. Let $s < t$, $s = (s_0, \ldots, s_{n-1})$, $t = (t_0, \ldots, t_{n-1})$ and $\tau^i = (s_0, \ldots, s_{i-1}, t_i, t_i, \ldots, t_{n-1})$. Then $\tau^0 = t$ and $\tau^n = s$. We may assume that $[\Delta_{n-1} h(y, v)]_t > 0$ for a particular $y$ and $v$ that we are considering. Thus

$$[\Delta_{n-1} h(y, v)]_{\tau^i} - [\Delta_{n-1} h(y, v)]_{\tau^{i-1}} = (s_i - t_i)[\Delta_{n} h(y, v)](s_0, \ldots, s_i, t_i, \ldots, t_{n-1}) > 0 \quad \text{for all } i, 0 < i < n.$$
If \( t_i = s_i \), the right side is not defined, but the left side is trivially 0. This proves (a).

(b)(i) From (2.1) we immediately see that the specific values
\[
\left[ \Delta_{n-1} h(y, v) \right](- (\delta/4)(1 + 1/(n - 1)), \ldots, (\delta/4)(1 + 1))
\]
and
\[
\left[ \Delta_{n-1} h(y, v) \right]((\delta/4)(1 + 1/(n - 1)), \ldots, (\delta/4)(1 + 1))
\]
are uniformly bounded (by, say, \( B_{n-1} \)) if \( y \in N_{\delta/2} \) and \( \|v\| < 1 \). Thus, by monotonicity, if \( |t_i| < \delta/2 \), \( \left[ \Delta_{n-1} h(y, v) \right] t \) is bounded by the same number.

(b)(ii) By downward induction on \( k \), we prove that \( \left[ \Delta_{k} h(y, v) \right] t \) is uniformly continuous and so uniformly bounded for \( y \in N_{\delta/2}, \|v\| < 1 \) and \( |t_j| < \delta/4, j = 0, 1, \ldots, k \). Now
\[
\left[ \Delta_{k} h(y, v) \right] (t_0, \ldots, t_{i-1}, t', t_{i+1}, \ldots, t_k)
\]
\[
- \left[ \Delta_{k} h(y, v) \right] (t_0, \ldots, t_{i-1}, t'', t_{i+1}, \ldots, t_k)
\]
\[
= (t' - t'') \left[ \Delta_{k+1} h(y, v) \right] (t_0, \ldots, t_{i-1}, t', t'', t_{i+1}, \ldots, t_k).
\]
We get a small change in values of \( \Delta_{k} h(y, v) \) when only one coordinate \( t \) changes by a small number. A general uniform continuity involves a sum of \((k - 1)\) such differences.

As an immediate consequence we get the following theorem.

**Theorem 3.2.** If a function \( f : A \to \mathbb{R} \) is \( n \)-convex \((n > 2)\) and locally bounded where \( A \) is any open subset of a Banach space \( E \), then

(i) For \( 0 < k < n - 2 \), the \( k \)-discrete differential \( \left( \hat{f}^{(k)} y \right) \) exists and is bounded for every \( y \in A \).

(ii) The one-sided \((n - 1)\)-discrete differential \( \left( \hat{f}^{(n-1)} y \right) \) exists and is bounded for every \( y \in A \).

(iii) \( f \) is uniformly continuous in a neighbourhood of every point \( y \in A \).

**Theorem 3.3.** Let \( A \) be an open subset of a Banach space \( E \). Let \( f : A \to \mathbb{R} \) be \( n \)-convex, \( n > 2 \), and bounded in \( N_{\delta} \), the \( \delta \)-neighbourhood of \( y_0 \in A \). Then for every \( \varepsilon > 0 \) and every sequence \( t = (t_0, \ldots, t_k) \) of distinct numbers such that \( |t_i| < \delta/4 \), there is a number \( \delta' > 0 \) such that
\[
\left| \left[ \Delta_{k} h(y', v) \right] t - \left[ \Delta_{k} h(y'', v) \right] t \right| < \varepsilon
\]
whenever \( \|y' - y_0\| < \delta/4, \|y'' - y_0\| < \delta/4, \|y'' - y'\| < \delta' \) and \( \|v\| < 1 \).

**Proof.** By (2.1),
\[
\left[ \Delta_{k} h(y', v) \right] t - \left[ \Delta_{k} h(y'', v) \right] t
\]
\[
= \frac{1}{\prod_{i \in [0,k]} (t_j - t_i)} \cdot \left[ f(y' + t_j v) - f(y'' + t_j v) \right].
\]
Since \( f \) is uniformly continuous in \( N_{\delta/2} \) by Theorem 3.2(iii), we have the result immediately.
**Theorem 3.4.** Let $A$ be any open subset of a Banach space $E$. If $f: A \to \mathbb{R}$ is $n$-convex, $n \geq 2$, and bounded in a neighbourhood $N_\delta$ of $y_0$ then for every $\varepsilon > 0$, there is a number $\delta' > 0$ such that

$$\left| \left( \hat{f}^k y' \right)_v - \left( \hat{f}^k y'' \right)_v \right| < \varepsilon$$

whenever $k = 0, 1, \ldots, n - 2$, $y', y'' \in N_{\delta/4}$, $\|y' - y''\| < \delta'$ and $\|v\| < 1$.

**Proof.** By strong uniform continuity of $\Delta_k h(y, v)$, the limit (2.2) is uniform (cf. Theorem 3.2(i)) in the sense that there is a number $\delta'' > 0$ such that

$$\left| \left( \hat{f}^k y \right)_v - (k!)\left[ \Delta_k h(y, v) \right]_t \right| < \varepsilon/3$$

whenever $y \in N_{\delta/4}$, $\|v\| < 1$ and $t = (t_0, \ldots, t_k)$ is a sequence of distinct numbers such that $0 < t_i < \delta''$. Then $(\hat{f}^k y')_v$ and $(\hat{f}^k y'')_v$ can be approximated uniformly by $(k!)\left[ \Delta_k h(y', v) \right]_t$ and $(k!)\left[ \Delta_k h(y'', v) \right]_t$, respectively using a fixed sequence of $t$'s. The result now follows from Theorem 3.3.

**Theorem 3.5.** If $A$ is an open subset of a Banach space $E$, $f: A \to \mathbb{R}$ is $n$-convex, $n \geq 3$, and $f$ is bounded in a neighbourhood $N_\delta$ of $y_0$. Then given $\varepsilon > 0$ there is a $\tilde{\delta} > 0$ such that

$$\left| \left[ \Delta_1 h(y', v) \right](0, t) - \left[ \Delta_1 h(y'', v) \right](0, t) \right| < \varepsilon$$

whenever $\|y' - y_0\| < \delta/4$, $\|y'' - y_0\| < \delta/4$, $\|y' - y''\| < \tilde{\delta}$, $t$ a real number with $0 < t < \tilde{\delta}$ and $\|v\| < 1$.

**Proof.** For any real number $s > t$ we have the identity

$$\left[ \Delta_1 h(y', v) \right](0, t) - \left[ \Delta_1 h(y'', v) \right](0, t) =$$

$$= \left[ \Delta_1 h(y', v) \right](0, s) - \left[ \Delta_1 h(y'', v) \right](0, s)$$

$$- (s - t)\left[ \Delta_2 h(y', v) \right](0, t, s) + (s - t)\left[ \Delta_2 h(y'', v) \right](0, t, s). \quad (3.3)$$

We may choose $s$ first small enough so that the values of $\Delta_2 h(y', v)$ and $\Delta_2 h(y'', v)$ are uniformly bounded by Theorem 3.1, then choose $s$ still smaller so that the absolute value of the $\Delta_2 h$ terms on the right side of (3.3) are less than or equal to $\varepsilon/4$, provided $0 < t < s$, $\|y' - y''\| < \delta/4$, $\|y' - y_0\| < \delta/4$ and $\|v\| < 1$. With a fixed choice of $s$, we may make the difference $y' - y''$ sufficiently small to make the $\Delta_1 h$ terms on the right side of (3.3) less than $\varepsilon/4$. Using Theorem 3.2, the number $\tilde{\delta}$ is chosen to insure that $\|y' - y''\|$ is small enough and that $t < s$. Q.E.D.

**4. Applications.** The results of §3 give that an $n$-convex locally bounded function has a $k$-discrete bounded differential, $1 < k < n - 2$, at every point of the domain and that a one-sided $(n - 1)$-discrete differential also exists and is bounded. This leads to the fact (proved in [4], and a detailed proof will be given elsewhere) that an $n$-convex locally bounded function admits a strong $(n - 2)$-Taylor series expansion (for definition, see [5]). With the aid of a local representation theorem (see Theorem 3.2 in [5]) it follows that the $k$th Fréchet differential, $1 < k < n - 2$, of an $n$-convex locally bounded function exists at every point of the domain. Later, this fact together with the assumption of a strong $n$-Taylor series expansion of $f$ at

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a point leads to \((n - 1)\)th order Fréchet differentiability at that point. These results are proved by the author in [4] and the detailed proof will appear elsewhere.

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