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A NOTE ON DISCONJUGATE DIFFERENTIAL EQUATIONS AND GROWTH ESTIMATES

L. ERBE[1]

ABSTRACT. Subfunctions and differential inequality techniques are applied to certain classes of second and third order nonlinear equations to obtain growth estimates on the solutions.

1. In several recent papers **[1]**, **[2]**, **[18]**, certain asymptotic properties, disconjugacy, boundedness, and oscillatory behavior have been studied for the equation

$$L[x](t) \equiv x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, x(t), \ldots, x^{(n-1)}(t))x^{(i)}(t) = 0 \tag{1}$$

where $a_i(t, v_1, \ldots, v_n)$ $i = 0, \ldots, n - 1$, are continuous on $I \times R^n$ and $I \subset R$ is an interval. References **[13]**, **[14]**, **[3]**, **[5]** contain results obtained for the analogous first order system

$$x' = A(t, x)x + f(t, x) \tag{2}$$

where $A(t, x)$ is an $n \times n$ matrix valued function defined on $I \times R^n$, and $f(t, x)$ and x are n-vectors. The results in **[1]**, **[2]**, **[18]** were obtained by considering the related linear equation

$$L_u[x](t) \equiv x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, u(t), \ldots, u^{(n-1)}(t))x^{(i)}(t) = 0 \tag{3}$$

for $u \in C^{(n-1)}(I)$. In particular, a nice application of a fixed-point theorem for multivalued mappings applied to solutions of (3) for $u \in B \subset C^{(n-1)}(I)$, B a bounded closed, convex subset, yields results for equation (1) analogous to those obtained for the linear nth order differential equation. In this note we apply subfunction and differential inequality techniques to obtain growth estimates on solutions of (1). For this reason, we restrict attention to the cases $n = 2, 3$ for which the required theory has been developed (see **[16]** and the references therein). Thus, we shall consider the equations

$$x'' + a_1(t, x, x')x' + a_0(t, x, x')x = 0 \tag{4}$$

and

$$x''' + a_2(t, x, x', x'')x'' + a_1(t, x, x', x'')x' + a_0(t, x, x', x'')x = 0. \tag{5}$$

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In addition to the continuity assumptions already mentioned, we shall assume that for equation (4) the following conditions hold.

(A_2) If x_1 and x_2 are solutions of equation (4) on $[t_1, t_2] \subset I$ where $t_1 < t_2$ such that $x_1(t_i) = x_2(t_i)$, $i = 1, 2$, then $x_1(t) \equiv x_2(t)$ on $[t_1, t_2]$.

(B_2) All solutions of all initial value problems for (4) extend throughout I.

For equation (5) the corresponding assumptions are

(A_3) If x_1 and x_2 are solutions of equation (5) on $[t_1, t_2] \subset I$ where $t_1 < t_2$ such that $x_1(t_i) = x_2(t_i)$, $i = 1, 2$, and either $x_1'(t_1) = x_2'(t_1)$ or $x_1'(t_2) = x_2'(t_2)$ then $x_1(t) \equiv x_2(t)$ on $[t_1, t_3]$.

(B_3) All solutions of all initial value problems for (5) extend throughout I.

We note that if (4) (or (5)) is linear then condition (A_2) (or (A_3)) is equivalent to disconjugacy of (4) (or (5)) on I and (B_2) (or (B_3)) holds trivially. It is known **[17]** that if $\phi \in C^2(I)$ is a lower solution of (4) on I; i.e.,

$$\phi'' + a_1(t, \phi, \phi')\phi' + a_0(t, \phi, \phi')\phi \geqslant 0 \tag{6}$$

and if (A_2) and (B_2) hold, then ϕ is a subfunction with respect to equation (4) on I. Similarly, if $\psi \in C^2(I)$ is an upper solution on I; i.e.,

$$\psi'' + a_1(t, \psi, \psi')\psi' + a_0(t, \psi, \psi')\psi \leqslant 0 \tag{7}$$

with (A_2) and (B_2) holding, then ψ is a superfunction with respect to equation (4) on I. For the third order case, we need an additional assumption which is:

(C_3) Either $f(t, x, x', x'') \equiv -\sum_{i=0}^{2} a_i(t, x, x', x'')x^{(i)}$ in (5) satisfies a Lipschitz condition with respect to x, x', and x'' on compact subsets of $I \times R^3$ or else $\phi, \psi \in C^3(I)$ are *strict* lower and upper solutions, respectively, for (5) on I.

We then have the result **[8]**, **[11]**. If $\phi \in C^3(I)$ is a lower solution for (5) (i.e., $\phi''' + \sum_{i=0}^{2} a_i(t, \phi, \phi', \phi'')\phi^{(i)} \geqslant 0$) and if ($A_3$), ($B_3$) and ($C_3$) hold, then ϕ is a subfunction with respect to (5) on I. Similarly, if $\psi \in C^3(I)$ is an upper solution for (5) (i.e., $\psi''' + \sum_{i=0}^{2} a_i(t, \psi, \psi', \psi'')\psi^{(i)} \leqslant 0$) and if ($A_3$), ($B_3$), ($C_3$) hold, then ψ is a superfunction with respect to (5) on I.

We note also that for equation (4) assumptions (A_2) and (B_2) imply that all two point boundary value problems on I do have solutions **[12]**. That is, for any t_1, $t_2 \in I$, $t_1 < t_2$, and any $c, d \in R$ there is a (unique) solution to (4) satisfying $y(t_1) = c$, $y(t_2) = d$. Similarly, assumptions (A_3) and (B_3) imply that all two point *and* three point boundary value problems for equation (5) have unique solutions **[8**, Theorem 2, p. 435]. As in **[16]** we shall assume that lower solutions are greater than or equal to upper solutions which is opposite to that usually found in papers dealing with second and third order equations.

2. The first two results are restatements or slight extensions of results of Schrader **[16]** and, since they do not depend on the particular form of the equation, are valid for the more general equations

$$x'' = f(t, x, x') \quad \text{and} \quad x''' = f(t, x, x', x'').$$

THEOREM 2.1. (a) *Let $I \subset R$ be an interval and assume that* (A_2) *and* (B_2) *hold for equation* (4). *Let $\phi, \psi \in C^2(I)$ be lower and upper solutions, respectively, of* (4) *on I*

with $\psi(t) \leqslant \phi(t)$ *on* I. *Then there is a solution* x *of* (4) *on* I *with* $\psi(t) \leqslant x(t) \leqslant \phi(t)$ *on* I.

(b) *Further, if* $I = [a, b]$ *is compact and* $\psi(t) < \phi(t)$ *on* $[a, b]$ *then there exist two distinct solutions* x, y *of* (4) *with* $\psi(t) \leqslant x(t) < y(t) \leqslant \phi(t)$ *on* $[a, b]$.

PROOF. Part (a) is Theorem 2.2 of **[16]**. So suppose $I = [a, b]$ and $\psi(t) < \phi(t)$ on $[a, b]$. It suffices to show that there exists a solution x of (4) with $\psi(t) \leqslant x(t) < \phi(t)$ on $[a, b]$ for then by an analogous argument there will exist a solution y of (4) with $x(t) < y(t) \leqslant \phi(t)$ on $[a, b]$. Following Schrader **[16**, Theorem 2.2, p. 207] for each $\lambda \in [a, b]$ let $\{v_n\}$ and $\{y_n\}$ be sequences of solutions of (4) on $[a, b]$ satisfying the respective initial conditions

$$v_n(\lambda) = \psi(\lambda), \qquad v_n'(\lambda) = \psi'(\lambda) + 1/n \tag{8}$$

and

$$y_n(\lambda) = \psi(\lambda), \qquad y_n'(\lambda) = \psi'(\lambda) - 1/n. \tag{9}$$

It follows as in **[16]** that there exist solutions v and y of (4) on $[a, b]$ which are limits of subsequences of $\{v_n\}$ and $\{y_n\}$, respectively, with $v(\lambda) = \psi(\lambda)$, $v'(\lambda) = \psi'(\lambda)$, $v(t) \geqslant \psi(t)$ for $\lambda \leqslant t \leqslant b$ and $y(\lambda) = \psi(\lambda)$, $y'(\lambda) = \psi'(\lambda)$, $y(t) \geqslant \psi(t)$ on $a \leqslant t \leqslant \lambda$. Thus the solution x_λ of (4) defined by $x_\lambda(t) = y(t)$ for $a \leqslant t \leqslant \lambda$ and $x_\lambda(t) = v(t)$ for $\lambda \leqslant t \leqslant b$ cannot (by **[17**, Theorem 1, p. 1007]) be equal to $\phi(t)$ at two points $t_1(\lambda)$, $t_2(\lambda) \in [a, b]$ with $t_1(\lambda) < \lambda < t_2(\lambda)$. We claim that there exists $\lambda_0 \in [a, b]$ such that $\psi(t) \leqslant x_{\lambda_0}(t) < \phi(t)$ on $[a, b]$. If not, then for each $\lambda \in [a, b]$ either $t_1(\lambda) < \lambda$ exists with $x_\lambda(t_1(\lambda)) = \phi(t_1(\lambda))$ or else $t_2(\lambda) > \lambda$ exists with $x_\lambda(t_2(\lambda)) = \phi(t_2(\lambda))$. Let $T_1 = \{\lambda \in [a, b]\colon t_1(\lambda) \text{ exists}\}$ and $T_2 = \{\lambda \in [a, b]\colon t_2(\lambda) \text{ exists}\}$. Since $t_1(b) \in T_1$ and $t_2(a) \in T_2$ neither T_1 nor T_2 is empty and $T_1 \cap T_2 = \varnothing$, $T_1 \cup T_2 = [a, b]$. Suppose $\{\lambda_n\} \subset T_1$ is a sequence and $\lambda_n \to \hat{\lambda}$. Then $\{t_1(\lambda_n)\} \subset [a, b]$ and there exists $x_n = x_{\lambda_n}$ a sequence of solutions of (4) with $x_n(t_1(\lambda_n)) = \phi(t_1(\lambda_n))$. There exists (by **[7**, Theorem 3, p. 14]) a subsequence of $\{x_n\}$ and $\{t_1(\lambda_n)\}$ which we again label $\{x_n\}$ and $\{t_1(\lambda_n)\}$ such that $t_1(\lambda_n) \to \hat{t}$ and x_n converges (uniformly on $[a, b]$) to a solution $\hat{x}$ of (4). Thus $\hat{x}(\hat{t}) = \lim_{n\to\infty} x_n(t_1(\lambda_n)) = \lim_{n\to\infty} \phi(t_1(\lambda_n)) = \phi(\hat{t})$ so that $\hat{t} = \hat{t}_1(\hat{\lambda})$ and $\hat{x} = x_{\hat{\lambda}}$. Hence, T_1 is closed so T_2 is open. Similarly, T_2 is closed. This contradiction shows that there exists $\lambda_0 \in [a, b] \setminus (T_1 \cup T_2)$ and therefore $\psi(t) \leqslant x_{\lambda_0}(t) < \phi(t)$ on $[a, b]$, as claimed. Now by an analogous argument with ϕ replacing ψ in conditions (8) and (9) we conclude the existence of a solution y_{λ_1} of (4) with $x_{\lambda_0}(t) < y_{\lambda_1}(t) \leqslant \phi(t)$ on $[a, b]$. This completes the proof.

For completeness we include the following

THEOREM 2.2 (SCHRADER **[16]**). *Let* $I \subset R$ *be an interval,* $\alpha \in I^0$ *and assume that* (A_3), (B_3) *hold. Let* $\phi, \psi \in C^3(I)$ *be lower and upper solutions, respectively, for* (5) *on* I *with* $\psi(t) \geqslant \phi(t)$ *on* $(-\infty, \alpha] \cap I$ *and* $\psi(t) \leqslant \phi(t)$ *on* $[\alpha, +\infty) \cap I$. *If* (C_3) *holds then there is a solution* x *of* (5) *on* I *with* $\psi(t) \geqslant x(t) \geqslant \phi(t)$ *on* $(-\infty, \alpha] \cap I$ *and* $\psi(t) \leqslant x(t) \leqslant \phi(t)$ *on* $[\alpha, +\infty) \cap I$.

PROOF. See Theorem 3.6 of **[16]**.

COROLLARY 2.3. *Let $I \subset R$ be an interval let* (A_3), (B_3) *hold, and for each $n \geqslant 1$ assume that ϕ_n, ψ_n are lower and upper solutions, respectively of* (5) *on I with* (C_3) *holding and with $\psi_n(t) \geqslant \phi_n(t)$ for $t \in (-\infty, \alpha_n) \cap I$ and $\psi_n(t) \leqslant \phi_n(t)$ for $t \in [\alpha_n, +\infty) \cap I$ where $\alpha_n \in I^0$. Assume further that ϕ_n, ψ_n are uniformly bounded on each compact subinterval of I and that* $\lim_{n\to\infty} \phi_n(t) \equiv \hat{\phi}(t)$ *and* $\lim_{n\to\infty} \psi_n(t) \equiv \hat{\psi}(t)$ *exist uniformly on compact subintervals of I and that* $\lim_{n\to\infty} \alpha_n = \hat{\alpha}$ ($\hat{\alpha} = \pm\infty$ *is allowed*). *Then there exists a solution x of* (5) *such that $\hat{\phi} \leqslant x \leqslant \hat{\psi}$ on $(-\infty, \hat{\alpha}] \cap I$ and $\hat{\psi} \leqslant x \leqslant \hat{\phi}$ on $[\hat{\alpha}, \infty) \cap I$.*

PROOF. For each $n \geqslant 1$ let x_n satisfy the conclusion of Theorem 2.2. Then since $\{x_n\}$ is a uniformly bounded sequence on each compact subinterval of I it follows by [**12**, Theorem 1] that there exists a subsequence of $\{x_n\}$ and a solution x of (5) such that $\lim x_n^{(i)}(t) = x^{(i)}(t)$, $i = 0, 1, 2$, uniformly on compact subintervals. If $\hat{\alpha} = +\infty$ then $\hat{\phi} \leqslant x \leqslant \hat{\psi}$ on I and if $\hat{\alpha} = -\infty$ then $\hat{\psi} \leqslant x \leqslant \hat{\phi}$ on I. If $\hat{\alpha} \in (-\infty, +\infty)$, then since $\phi_n \leqslant x \leqslant \psi_n$ on $(-\infty, \alpha_n] \cap I$ and $\psi_n \leqslant x_n \leqslant \phi_n$ on $[\alpha_n, +\infty) \cap I$, the conclusion follows.

3. Examples and applications. We shall now illustrate the applicability of the above subfunction-differential inequality technique to equations of the form (4) and (5). In particular, the results of [**1**], [**2**] do not apply in the following examples.

EXAMPLE 3.1. Let equation (4) satisfy (A_2) and (B_2) on I and assume there exist constants $\beta < \alpha$ with $a_0(t, \beta, 0)\beta \leqslant 0 \leqslant a_0(t, \alpha, 0)\alpha$. Then there exists a solution $x(t)$ of (4) with $\beta \leqslant x(t) \leqslant \alpha$ on I.

EXAMPLE 3.2. Let equation (4) satisfy (A_2) and (B_2) and assume there exists $\lambda > 0$ such that $\lambda^2 + a_1(t, x, x')\lambda + a_0(t, x, x') \geqslant 0$ for all $|x| + |x'| < +\infty$ and $t \in I = [a, b]$. Then there exist solutions $x(t), y(t)$ of (4) such that $0 \leqslant x(t) < y(t) \leqslant e^{\lambda t}$ on $[a, b]$. (Here $\psi(t) \equiv 0$ and $\phi(t) \equiv e^{\lambda t}$ are upper and lower solutions, respectively, so Theorem 2.1b yields the result.)

EXAMPLE 3.3. Let there exist $0 < \mu < 1 < \nu$ such that $-\nu(\nu - 1) \leqslant t^2p(t) \leqslant \mu(1 - \mu)$ on $[a, +\infty)$, $a > 0$. The Sturm Comparison Theorem [**19**] shows then that the linear equation

$$x'' + p(t)x = 0 \tag{10}$$

is disconjugate on $[a, +\infty)$ (by comparison with the disconjugate Euler equation $x'' + \frac{1}{4}t^{-2}x = 0$). Therefore from Theorem 2.1 (with $\psi(t) \equiv t^{\mu} < t^{\nu} \equiv \phi(t)$) we conclude the existence of a solution $x(t)$ of (10) with $t^{\mu} \leqslant x(t) \leqslant t^{\nu}$. Further, from results of [**9**, Theorem 7.4] there also exists a solution $y(t)$ of (10) with $0 < y(t) \leqslant t^{\mu}$ on $[a, +\infty)$.

We conclude with several examples for the case $n = 3$.

EXAMPLE 3.4. Consider the third order linear equation

$$x''' + p(t)x = 0 \tag{11}$$

and assume that $|t^3p(t)| \leqslant 2/3\sqrt{3}$ for $t \geqslant a$ where $0 < a < 1$. From [**19**, p. 163] it follows (by comparison with the disconjugate Euler equations $x''' \pm 2t^{-3}x/3\sqrt{3} = 0$ on $[a, +\infty)$, $a > 0$) that (11) is disconjugate on $[a, +\infty)$ so that (A_3), (B_3), (C_3)

hold. With $\mu = 1 + \sqrt{3}/3$ and $\nu > 2$ the unique solution of $\nu(\nu - 1)(\nu - 2) = 2/3\sqrt{3}$ it follows that $\phi(t) \equiv t^\nu$ and $\psi(t) \equiv t^\nu$ are lower and upper solutions on $[a, +\infty)$. Therefore Theorem 2.2 implies the existence of a solution $x(t)$ of (11) satisfying $t^\mu \leqslant x(t) \leqslant t^\nu$ on $[1, +\infty)$ and $t^\nu \leqslant x(t) \leqslant t^\mu$ on $[a, 1]$. We note that the results of **[2]** (see also **[10]**) concerning the separation of the roots of the characteristic equation $\lambda^3 + p(t) = 0$ by constants is never fulfilled for an equation of the form (11) so that those techniques are not applicable here.

EXAMPLE 3.5. Consider equation

$$x''' + a_2(t, x, x')x'' + a_1(t, x, x')x' + a_0(t, x, x')x = 0 \tag{5}$$

and suppose conditions (A_3), (B_3) hold on $I = [a, b]$ and let there exist constants $\rho_1 < \rho_2$ with

$$\rho_1^3 + a_2(t, x, x')\rho_1^2 + a_1(t, x, x')\rho_1 + a_0(t, x, x') < 0 \tag{12}$$

and

$$\rho_2^3 + a_2(t, x, x')\rho_2^2 + a_1(t, x, x')\rho_2 + a_0(t, x, x') > 0 \tag{13}$$

for all $|x| + |x'| < +\infty$. Then it follows that for any $a \leqslant t_0 \leqslant b$, $\psi_{t_0}(t) \equiv e^{(t-t_0)\rho_1}$ and $\phi_{t_0}(t) \equiv e^{(t-t_0)\rho_2}$ are upper and lower solutions for (5) on $[a, b]$. Thus by Theorem 2.2 (or Corollary 2.3) there exists a solution $x(t; t_0)$ of (5) with $\phi_{t_0}(t) \leqslant x(t; t_0) \leqslant \psi_{t_0}(t)$ on $[a, t_0]$ and $\psi_{t_0}(t) \leqslant x(t; t_0) \leqslant \phi_{t_0}(t)$ on $[t_0, b]$. This result may not be concluded from the results of **[1]**, **[2]**. We note that if $a_i(t, x, x')$ are bounded on $[a, b] \times R^2$ then (12) and (13) can always be satisfied for sufficiently large negative ρ_1 and large positive ρ_2.

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Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2H1