ON THE ANALYTIC COHOMOLOGY OF
A DOMAIN IN A STEIN MANIFOLD

MICHAEL G. EASTWOOD

Abstract. Suppose $M$ is an open subset of a Stein manifold without isolated points and that $V$ is a holomorphic vector-bundle on $M$ admitting a holomorphic connection. Then $H^q(M, \Theta(V))$ is either zero or of infinite dimension.

In [1] Laufer showed that, for $M$ an open subset of a Stein manifold without isolated points, $H^q(M, \Omega^p)$ is either zero or of infinite dimension as a complex vector-space. Here $\Omega^p$ denotes the sheaf of germs of holomorphic $p$-forms. In this note we extend this result to $H^q(M, \Theta(V))$ for any holomorphic vector-bundle $V$ obtained as the restriction of a holomorphic vector-bundle on the ambient Stein manifold (see corollary below). The argument is an extension of that given by Laufer for $H^q(M, \Theta)$. The crucial ingredient is a connection on $V$ (see theorem below).

Since making this extension I have learned that these results were also obtained by John Sensat, a student of Professor Laufer. Unfortunately, he was killed in an accident approximately three years ago. I respectfully dedicate this paper to his memory.

Lemma. Suppose $V$ is a holomorphic vector-bundle on a complex manifold $M$. Suppose $V$ admits a holomorphic connection. Fix a nonnegative integer $q$ and let $J$ denote the ideal of holomorphic functions on $M$ which induce (by multiplication on the sheaf level) the zero map of $H^q(M, \Theta(V))$ to itself. Then $f \in J \Rightarrow Xf \in J$ for any holomorphic vector-field $X$ on $M$.

Proof. Let $\nabla: \Theta(V) \to \Omega^1(V)$ be a connection. By the Leibnitz rule $(Xf)s = \nabla_X(fs) - f\nabla_Xs$ for $s \in \Theta(V)$. By functoriality the same is true for $s \in H^q(M, \Theta(V))$ and the result follows.

Theorem. Suppose $M$ is an open subset of a Stein manifold without isolated points. Suppose $V$ is a holomorphic vector-bundle on $M$ that admits a holomorphic connection. Then $H^q(M, \Theta(V))$ is either zero or of infinite dimension.

Proof. Denote by $S$ the ambient Stein manifold and regard $S$ as being embedded in $\mathbb{C}^N$. Let $I$ denote the ideal of holomorphic functions on $S$ which induce the zero map on $H^q(M, \Theta(V))$, i.e., using the notation of the lemma above, $f \in I \Rightarrow f|_M \in J$. Suppose $H^q(M, \Theta(V))$ is of finite dimension. It suffices to show

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that $1 \in I$. The coordinate functions $z_j$ on $\mathbb{C}^N$ induce linear endomorphisms of $H^q(M, \Theta(\mathcal{V}))$. Let $p_j$ be the minimal polynomial of $z_j$ regarded as such an endomorphism. Then $p_j(z_j) \in I$ and the collection $p_1(z_1), \ldots, p_N(z_N)$ have only finitely many common zeros on $S$. However, for any $x \in S$ we can choose $f \in I$ with $f(x) \neq 0$, for first choose any nonzero $f \in I$ and then use the lemma above repeatedly to reduce the order of the zero. Here we are using Cartan's Theorem B to assert that we can specify the value at $x$ of a holomorphic vector-field on $S$. Thus we can augment our collection $p_j(z_j)$ to a collection, say $f_1, \ldots, f_n \in I$, with no common zeros. Now Theorem B shows that we can find $g_1, \ldots, g_n \in \Gamma(S, \Theta)$ with $\sum g_j f_j = 1$ and so $1 \in I$ as required.

**Corollary.** Suppose $M$ is an open subset of a Stein manifold $S$ without isolated points. Suppose $V$ is a holomorphic vector-bundle on $S$. Then $H^q(M, \Theta(\mathcal{V}))$ is either zero or of infinite dimension.

**Proof.** $V$ admits a holomorphic connection on $S$ since the obstruction lies in $H^1(S, \Omega^1(\text{End } V))$ and this vanishes by Cartan's Theorem B. Now restrict to $M$ and apply the theorem.

**References**


School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540