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EXTENSION OF BERNSTEIN'S THEOREM

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ABSTRACT. A well-known theorem of Bernstein states that if a polynomial of degree n of a complex variable has its modulus no larger than one on the unit disk then the modulus of its derivative will not exceed n on the unit disk. Here we extend the theorem to polynomials on the unit ball in several complex variables.

Let $\mathbf{C}$ be the field of complex numbers and $\mathbf{C}^m$ the vector space over $\mathbf{C}$ of m-tuples. If $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$, we set $\|z\| = \sqrt{\Sigma_{i=1}^m |z_i|^2}$. The set $B_m = \{z: \|z\| \leqslant 1\}$ is called the unit ball in $\mathbf{C}^m$. For a polynomial P in $\mathbf{C}^m$, we set

$$DP(z) = (D_1P(z), \ldots, D_mP(z))$$

where $D_iP(z)$ denotes the partial derivative of P with respect to z_i at z for $i = 1, \ldots, m$. A function $H(z, w)$ defined for $z, w \in \mathbf{C}^m$ with values in $\mathbf{C}$ is called a hermitian symmetric form if $H(z, w)$ is linear in z for every fixed w and $H(w, z) = \overline{H(z, w)}$.

In this paper, we extend the following complex version of Bernstein's theorem [5, p. 57] from a polynomial on the unit disk to a polynomial on the unit ball in several complex variables. Similar extensions of Bernstein's theorem and Markoff's theorem for polynomials of several real variables were obtained by Kellogg [4].

BERNSTEIN'S THEOREM. *If P is a polynomial of degree n in $z \in \mathbf{C}$ with $|P(z)| \leqslant 1$ on $|z| \leqslant 1$, then $|P'(z)| \leqslant n$ for $|z| \leqslant 1$. This result is best possible and equality holds for $P(z) = \alpha z^n$ where $|\alpha| = 1$.*

First, we need the following lemmas.

LEMMA 1. *Let P be a polynomial of degree n in $z \in \mathbf{C}$. If $|P(z)| \leqslant 1$ for $|z| \leqslant r$ and $|P(z_0)| = 1$ where $|z_0| = r$, then* (i) $P'(z_0) \neq 0$ *and* (ii) $\arg z_0P'(z_0) = \arg P(z_0)$.

PROOF. Without loss of generality, we may assume that $z_0 = r$ and $P(r) = 1$. If $P'(r)$ were to be 0, then the Taylor expansion of P at r is

$$P(z) = P(r) + a_k(z - r)^k + a_{k+1}(z - r)^{k+1} + \ldots$$

with $a_k \neq 0$ for $k \geqslant 2$. This can be expressed as

$$P(z) - 1 = (z - r)^k[a_k + a_{k+1}(z - r) + \ldots].$$

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Taking the argument of both sides, we have

$$\arg(P(z) - 1) = k \arg(z - r) + \arg[a_k + a_{k+1}(z - r) + \dots].$$

Now, for $z = r + \varepsilon e^{i\theta}$ with ε sufficiently small, as θ increases from $\pi/2$ to $3\pi/2$ the increment of the argument on the left-hand side is no more than π because of $|P(z)| \leqslant 1$, while the increment on the right-hand side is nearly 2π or more. This proves that $P'(r) \neq 0$.

The assumptions of the lemma and (i) suggest that the image of the circle $|z| = r$ under the mapping P is a smooth curve tangent to the circle $|w| = 1$ at $P(z_0)$. Hence the mapping P is locally starlike in a neighborhood of z_0 with respect to the origin. Thus, from [**2**, p. 357], we have $\operatorname{Re}\{z_0P'(z_0)/P(z_0)\} > 0$. Also, since $|P(z)|$ attains its maximum $|P(z_0)| = 1$, we have $\operatorname{Im}\{z_0P'(z_0)/P(z_0)\} = 0$. Thus (ii) follows immediately.

Geometrically, (ii) means that the normal to the image of $|z| = r$ at $P(z_0)$ coincides with the normal to the circle $|w| = 1$.

LEMMA 2. *Let P be a homogeneous polynomial of degree n in $z \in \mathbf{C}^m$ and $|P(z)| \leqslant 1$ for $\|z\| \leqslant 1$. Suppose that $|P(a)| = 1$ where $\|a\| = 1$. Then $\sum_{i=1}^m |a_iD_i(a)| = n$.*

PROOF. P satisfies the Euler identity

$$\sum_{i=1}^m z_iD_iP(z) = nP(z) \quad \text{for every } z \in \mathbf{C}^m.$$

For each fixed i, $P(z)$ may be regarded as a polynomial of degree n in z_i. Hence, from Lemma 1, $\arg a_iD_iP(a) = \arg P(a)$ for $i = 1, \dots, m$. Therefore

$$\sum |a_iD_iP(a)| = \left|\sum a_iD_iP(a)\right| = |nP(a)| = n.$$

The following lemma is a complex version of a generalization of Laguerre's theorem due to Hörmander [**3**, p. 57], [**6**, p. 57].

LEMMA 3. *Given a homogeneous polynomial $P(z)$ of degree n and a hermitian symmetric form $H(z, w)$ defined in $\mathbf{C}^m$. Let $T_m = \{z\colon z \neq 0, H(z, z) > 0\}$. Suppose that $P(z) \neq 0$ for every $z \in T_m$. Then it follows that $\sum_{i=1}^m w_iD_iP(z) \neq 0$ when both $z, w \in T_m$.*

We obtain an extension of Bernstein's theorem from the unit disk in $\mathbf{C}$ to the unit ball in $\mathbf{C}^m$ as follows.

THEOREM. *Let P be a polynomial of degree n in $\mathbf{C}^k$ and $|P(z)| \leqslant 1$ for $\|z\| \leqslant 1$. Then*

$$\sum_{i=1}^k r_i|D_iP(z)| \leqslant n$$

where r_i real positive and $\sum_{i=1}^k r_i^2 = 1$. This result is best possible and equality holds for a given z_0, $\|z_0\| = 1$, when P is homogeneous and $|P(z_0)| = 1$.

PROOF. Since $|P(z)| \leqslant 1$ for $\|z\| < 1$, we have $P(z) + \lambda e^{i\theta} \neq 0$ for any real θ and $\lambda > 1$. If we set $z^* = (z_0, z)$ where $z = (z_1, \ldots, z_k)$, $z_0 \in \mathbf{C}$, and

$$f(z^*) = z_0^n\left[P(z/z_0) + \lambda e^{i\theta}\right] = P(z_0, z) + \lambda e^{i\theta} z_0^n,$$

then both $P(z_0, z)$ and $f(z^*)$ are homogeneous polynomials of degree n in $z^* = (z_0, z) \in \mathbf{C}^{k+1}$. Consider

$$H(z^*, z^*) = |z_0|^2 - \sum_{i=1}^{k} |z_i|^2.$$

By setting $|z_0| = 1$ we have $H(z^*, z^*) \geqslant 0$ for $\|z\| < 1$. That is, for $z, w \in B_k$, $z^*, w^* \in T_{k+1}$. Therefore, from Lemma 3, $f(z^*) \neq 0$ implies

$$\sum_{i=0}^{k} w_i D_i f(z^*) \neq 0.$$

This means

$$\sum_{i=0}^{k} w_i D_i P(z^*) \neq -n\lambda e^{i\theta} z_0^{n-1} w_0.$$

Since this inequality holds for any real θ and $\lambda > 1$, by setting $z_0 = w_0 = 1$ we have

$$\left|\sum_{i=0}^{k} w_i D_i P(z^*)\right| < n\lambda.$$

For each i, choose w_i such that $\arg \overline{w}_i = \arg D_i P(z^*)$, the above inequality becomes

$$\sum_{i=0}^{k} |w_i|\, |D_i P(z^*)| < n\lambda.$$

By dropping the first term, $i = 0$, of the left-hand side and making λ arbitrarily close to 1, we have

$$\sum_{i=1}^{k} |w_i|\, |D_i P(z)| \leqslant n \quad \text{for } \|z\| \leqslant 1, \|w\| \leqslant 1.$$

Here, if we set $r_i = |w_i| \geqslant 0$, the inequality in the theorem is obtained.

When P is homogeneous of degree n in $z \in \mathbf{C}^k$, from Lemma 2 the equality holds for the theorem since, when $|P(z_0)| = 1$ with $\|z_0\| = 1$, we have

$$\sum |z_i D_i P(z_0)| = \left|\sum z_i D_i P(z_0)\right| = |nP(z_0)| = n.$$

This theorem can also be stated in the following form.

COROLLARY. *Let P be a polynomial of degree n in $\mathbf{C}^k$ and $|P(z)| \leqslant 1$ for $\|z\| \leqslant 1$. Then $\|DP(z)\| \leqslant n$ for $\|z\| \leqslant 1$.*

PROOF. Since $\Sigma|w_i|\, |D_i P(z)| \leqslant n$ for $\|z\| \leqslant 1$, $\|w\| \leqslant 1$, by Cauchy's inequality [**1**, p. 10]

$$\max_{\|w\| \leqslant 1} \sum_{i=1}^{k} |w_i|\, |D_i P(z)| = \sqrt{\sum_{i=1}^{k} |D_i P(z)|^2} = \|DP(z)\|.$$

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