EXTENSION OF BERNSTEIN'S THEOREM

S. H. TUNG

ABSTRACT. A well-known theorem of Bernstein states that if a polynomial of degree \( n \) of a complex variable has its modulus no larger than one on the unit disk then the modulus of its derivative will not exceed \( n \) on the unit disk. Here we extend the theorem to polynomials on the unit ball in several complex variables.

Let \( \mathbb{C} \) be the field of complex numbers and \( \mathbb{C}^m \) the vector space over \( \mathbb{C} \) of \( m \)-tuples. If \( z = (z_1, \ldots, z_m) \in \mathbb{C}^m \), we set \( \|z\| = \sqrt{\sum_{i=1}^{m}|z_i|^2} \). The set \( B_m = \{z: \|z\| < 1\} \) is called the unit ball in \( \mathbb{C}^m \). For a polynomial \( P \) in \( \mathbb{C}^m \), we set

\[
DP(z) = (D_1P(z), \ldots, D_mP(z))
\]

where \( D_iP(z) \) denotes the partial derivative of \( P \) with respect to \( z_i \) at \( z \) for \( i = 1, \ldots, m \). A function \( H(z, w) \) defined for \( z, w \in \mathbb{C}^m \) with values in \( \mathbb{C} \) is called a hermitian symmetric form if \( H(z, w) \) is linear in \( z \) for every fixed \( w \) and \( H(w, z) = H(z, w) \).

In this paper, we extend the following complex version of Bernstein's theorem [5, p. 57] from a polynomial on the unit disk to a polynomial on the unit ball in several complex variables. Similar extensions of Bernstein's theorem and Markoff's theorem for polynomials of several real variables were obtained by Kellogg [4].

Bernstein's theorem. If \( P \) is a polynomial of degree \( n \) in \( z \in \mathbb{C} \) with \( |P(z)| < 1 \) on \( |z| < 1 \), then \( |P'(z)| < n \) for \( |z| < 1 \). This result is best possible and equality holds for \( P(z) = az^n \) where \( |a| = 1 \).

First, we need the following lemmas.

Lemma 1. Let \( P \) be a polynomial of degree \( n \) in \( z \in \mathbb{C} \). If \( |P(z)| < 1 \) for \( |z| < r \) and \( |P(z_0)| = 1 \) where \( |z_0| = r \), then (i) \( P'(z_0) \neq 0 \) and (ii) \( \arg z_0P'(z_0) = \arg P(z_0) \).

Proof. Without loss of generality, we may assume that \( z_0 = r \) and \( P(r) = 1 \). If \( P'(r) \) were to be 0, then the Taylor expansion of \( P \) at \( r \) is

\[
P(z) = P(r) + \sum_{k=1} P^{(k-1)}(r)(z-r)^{k} + \ldots
\]

with \( a_k \neq 0 \) for \( k > 2 \). This can be expressed as

\[
P(z) - 1 = (z-r)^k [a_k + a_{k+1}(z-r) + \ldots].
\]
Taking the argument of both sides, we have
\[ \arg(P(z) - 1) = k \arg(z - r) + \arg[a_k + a_{k+1}(z - r) + \ldots]. \]

Now, for \( z = r + \epsilon e^{i\theta} \) with \( \epsilon \) sufficiently small, as \( \theta \) increases from \( \pi/2 \) to \( 3\pi/2 \) the increment of the argument on the left-hand side is no more than \( \pi \) because of \( |P(z)| < 1 \), while the increment on the right-hand side is nearly \( 2\pi \) or more. This proves that \( P'(r) \neq 0 \).

The assumptions of the lemma and (i) suggest that the image of the circle \( |z| = r \) under the mapping \( P \) is a smooth curve tangent to the circle \( |w| = 1 \) at \( P(z_0) \). Hence the mapping \( P \) is locally starlike in a neighborhood of \( z_0 \) with respect to the origin. Thus, from [2, p. 357], we have \( \Re\{z_0 P'(z_0)/P(z_0)\} > 0 \). Also, since \( |P(z)| \) attains its maximum \( |P(z_0)| = 1 \), we have \( \Im\{z_0 P'(z_0)/P(z_0)\} = 0 \). Thus (ii) follows immediately.

Geometrically, (ii) means that the normal to the image of \( |z| = r \) at \( P(z_0) \) coincides with the normal to the circle \( |w| = 1 \).

**Lemma 2.** Let \( P \) be a homogeneous polynomial of degree \( n \) in \( z \in \mathbb{C}^n \) and \( |P(z)| < 1 \) for \( \|z\| < 1 \). Suppose that \( |P(a)| = 1 \) where \( \|a\| = 1 \). Then \( \sum_{i=1}^{n} |a_i D_i P(a)| = n \).

**Proof.** \( P \) satisfies the Euler identity

\[ \sum_{i=1}^{m} z_i D_i P(z) = nP(z) \quad \text{for every } z \in \mathbb{C}^m. \]

For each fixed \( i \), \( P(z) \) may be regarded as a polynomial of degree \( n \) in \( z_i \). Hence, from Lemma 1, \( \arg a_i D_i P(a) = \arg P(a) \) for \( i = 1, \ldots, m \). Therefore

\[ \sum |a_i D_i P(a)| = \left| \sum a_i D_i P(a) \right| = |nP(a)| = n. \]

The following lemma is a complex version of a generalization of Laguerre’s theorem due to Hörmander [3, p. 57], [6, p. 57].

**Lemma 3.** Given a homogeneous polynomial \( P(z) \) of degree \( n \) and a hermitian symmetric form \( H(z, w) \) defined in \( \mathbb{C}^m \). Let \( T_m = \{ z : z \neq 0, H(z, z) > 0 \} \). Suppose that \( P(z) \neq 0 \) for every \( z \in T_m \). Then it follows that \( \sum_{i=1}^{m} w_i D_i P(z) \neq 0 \) when both \( z, w \in T_m \).

We obtain an extension of Bernstein’s theorem from the unit disk in \( \mathbb{C} \) to the unit ball in \( \mathbb{C}^m \) as follows.

**Theorem.** Let \( P \) be a polynomial of degree \( n \) in \( \mathbb{C}^k \) and \( |P(z)| < 1 \) for \( \|z\| < 1 \). Then

\[ \sum_{i=1}^{k} r_i |D_i P(z)| < n \]

where \( r_i \) real positive and \( \sum_{i=1}^{k} r_i^2 = 1 \). This result is best possible and equality holds for a given \( z_0, \|z_0\| = 1 \), when \( P \) is homogeneous and \( |P(z_0)| = 1 \).
Proof. Since \(|P(z)| < 1\) for \(||z|| < 1\), we have \(P(z) + \lambda e^{i\theta} \neq 0\) for any real \(\theta\) and \(\lambda > 1\). If we set \(z^* = (z_0, z)\) where \(z = (z_1, \ldots, z_k)\), \(z_0 \in \mathbb{C}\), and 
\[
f(z^*) = z_0^n \left[ P(z/z_0) + \lambda e^{i\theta} \right] = P(z_0, z) + \lambda e^{i\theta} z_0^n,
\]
then both \(P(z_0, z)\) and \(f(z^*)\) are homogeneous polynomials of degree \(n\) in \(z^* = (z_0, z) \in \mathbb{C}^{k+1}\). Consider
\[
H(z^*, z^*) = |z_0|^2 - \sum_{i=1}^{k} |z_i|^2.
\]
By setting \(|z_0| = 1\) we have \(H(z^*, z^*) > 0\) for \(||z|| < 1\). That is, for \(z, w \in B_k\), \(z^*, w^* \in T_{k+1}\). Therefore, from Lemma 3, \(f(z^*) \neq 0\) implies
\[
\sum_{i=0}^{k} w_i D_i f(z^*) \neq 0.
\]
This means
\[
\sum_{i=0}^{k} w_i D_i P(z^*) \neq -n\lambda e^{i\theta} z_0^{n-1} w_0.
\]
Since this inequality holds for any real \(\theta\) and \(\lambda > 1\), by setting \(z_0 = w_0 = 1\) we have
\[
\left| \sum_{i=0}^{k} w_i D_i P(z^*) \right| < n\lambda.
\]
For each \(i\), choose \(w_i\) such that \(\arg w_i = \arg D_i P(z^*)\), the above inequality becomes
\[
\sum_{i=0}^{k} |w_i| |D_i P(z^*)| < n\lambda.
\]
By dropping the first term, \(i = 0\), of the left-hand side and making \(\lambda\) arbitrarily close to 1, we have
\[
\sum_{i=1}^{k} |w_i| |D_i P(z)| < n \quad \text{for} \quad ||z|| < 1, ||w|| < 1.
\]
Here, if we set \(r_i = |w_i| > 0\), the inequality in the theorem is obtained.

When \(P\) is homogeneous of degree \(n\) in \(z \in \mathbb{C}^k\), from Lemma 2 the equality holds for the theorem since, when \(|P(z_0)| = 1\) with \(||z_0|| = 1\), we have
\[
\sum |z_i D_i P(z_0)| = |\sum z_i D_i P(z_0)| = |nP(z_0)| = n.
\]
This theorem can also be stated in the following form.

Corollary. Let \(P\) be a polynomial of degree \(n\) in \(\mathbb{C}^k\) and \(|P(z)| < 1\) for \(||z|| < 1\). Then \(||DP(z)|| < n\) for \(||z|| < 1\).

Proof. Since \(\sum |w_i| |D_i P(z)| < n\) for \(||z|| < 1\), \(||w|| < 1\), by Cauchy’s inequality [1, p. 10]
\[
\max_{||w|| < 1} \sum_{i=1}^{k} |w_i| |D_i P(z)| = \sqrt{\sum_{i=1}^{k} |D_i P(z)|^2} = ||DP(z)||.
\]
REFERENCES


Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056