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DERIVATIVES OF POLYNOMIALS WITH POSITIVE COEFFICIENTS

A. K. VARMA

ABSTRACT. Let $P_n(x)$ be an algebraic polynomial of degree n with positive coefficients. We set

$$I_n = \frac{\|P_n'(x)\omega(x)\|_{L_2[0,\infty)}}{\|P_n(x)\omega(x)\|_{L_2[0,\infty)}}.$$

In this work upper bounds of I_n are investigated. We restrict ourselves here with the case $\omega(x) = x^{\alpha/2}e^{-x/2}$. Results are shown to be best possible.

1. The problems dealt with in the present paper are connected with the classical inequalities of A. Markov [5], P. Erdos [1], G. G. Lorentz [3], [4], G. Szego [7] and P. Turan [8]. First, we describe them briefly.

THEOREM A (A. A. MARKOV). *If $f(x)$ is a polynomial of degree n, such that $|f(x)| \leqslant 1$ for $a \leqslant x \leqslant b$, then*

$$\max_{a \leqslant x \leqslant b} |f'(x)| \leqslant \frac{2}{b-a} n^2. \tag{1.1}$$

Further, this bound is sharp, as can be seen by the example

$$f(x) = T_n(2(x-a)/(b-a) - 1)$$

where T_n is Chebyshev's polynomial.

For special polynomials, one can sometimes improve the estimate (1.1).

Next, two theorems are of this kind.

THEOREM B (P. ERDOS). *Let $f(x)$ be a polynomial of degree n satisfying the inequality $|f(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$. Suppose $f(x)$ has only real roots and no root inside $[-1, +1]$; then for $-1 \leqslant x \leqslant 1$,*

$$|f'(x)| \leqslant \tfrac{1}{2} en. \tag{1.2}$$

This is the best possible result.

Inequalities of the same type (as in Theorem B) hold for the wider class of polynomials with positive coefficients in $1 + x$, $1 - x$; that is, for the polynomials

$$f(x) = \sum_{k+l \leqslant n} a_{kl}(1+x)^k(1-x)^l, \qquad a_{kl} \geqslant 0, \quad -1 \leqslant x \leqslant 1. \tag{1.3}$$

These polynomials were introduced by G. G. Lorentz [3] and studied extensively by J. T. Scheick [6]. The Lorentz theorem can be stated as follows.

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THEOREM C (G. G. LORENTZ). *There exists a constant $C > 0$ such that for each polynomial $f(x)$ of the form* (1.3),

$$\frac{\|f'\|}{\|f\|} \leqslant Cn, \qquad n = 1, 2, \ldots, \tag{1.4}$$

for the uniform norm on $[-1, +1]$.

In 1964 G. Szego [7] studied the order of magnitude of $\|f'\|/\|f\|$ for unrestricted polynomials f of degree $\leqslant n$ for the norm

$$\|f\| = \sup_{x \geqslant 0} |f(x)e^{-x}| \tag{1.5}$$

on $(0, \infty)$. More precisely, he proved the following.

THEOREM D (G. SZEGO). *Let $f(x)$ be a polynomial of fixed degree n and not vanishing identically. Then*

$$\frac{\|f'\|}{\|f\|} < Cn, \qquad n = 2, 3, \ldots, \tag{1.6}$$

holds (for the weighted uniform norm as defined by (1.5) *on* $(0, \infty)$).

In 1968 G. G. Lorentz [4] considered the problem of G. Szego for the special polynomials with positive coefficients in x,

$$f(x) = \sum_{k=0}^{n} a_k x^k, \qquad a_k \geqslant 0, \tag{1.7}$$

and the norm of a function f on $(0, \infty)$ is given by

$$\|f\| = \sup_{x \geqslant 0} |f(x)e^{-\omega(x)}|,$$

where ω increases on $(0, \infty)$. Then the Lorentz theorem can be stated as follows.

THEOREM E (G. G. LORENTZ). *Let $\omega(x)$ satisfy the inequalities $\omega(x) - \omega(0) \leqslant Ax\omega'(x)$, $x \geqslant 0$, and $\omega'(y) \leqslant A\omega'(x)$, $y \leqslant x$, for some constant $A > 0$. Then for some constant $C > 0$, the inequality*

$$\frac{\|f'\|}{\|f\|} \leqslant C\frac{\|p_n'\|}{\|p_n\|}, \qquad p_n(x) = x^n,$$

is valid.

Other related interesting results are due to A. Zygmund [10], Hille, Szego and Tamerkin [2] and P. Turan [8], where similar problems are considered either in L_2 norm or even in L_p norm.

We may note that Lorentz raised the problem of determining the best constant in (1.6) and for (1.9).

Now we state the main theorem of this paper. Let $S_n[a, b]$ be the set of all polynomials whose degree is n and whose roots are all real and no root inside (a, b). It is easy to see that if $p_n \in S_n[0, \infty)$ then it can be expressed in the form

$$P_n(x) = \sum_{k=0}^{n} a_k x^k, \qquad a_k \geqslant 0 \text{ for } k = 0, 1, \ldots, n. \tag{1.8}$$

We now state

THEOREM 1. *Let $P_n(x)$ be an algebraic polynomial of degree n with nonnegative coefficients. Then for $\alpha \geqslant (\sqrt{5} - 1)/2$*

$$\int_0^\infty (P_n'(x))^2 x^\alpha e^{-x}\, dx \leqslant \frac{n^2}{(2n+\alpha)(2n+\alpha-1)} \int_0^\infty P_n^2(x) x^\alpha e^{-x}\, dx, \tag{1.9}$$

equality holding for $P_n(x) = x^n$. For $0 \leqslant \alpha \leqslant \frac{1}{2}$ we have

$$\int_0^\infty (P_n'(x))^2 x^\alpha e^{-x}\, dx \leqslant \frac{1}{(2+\alpha)(1+\alpha)} \int_0^\infty P_n^2(x) x^\alpha e^{-x}\, dx. \tag{1.10}$$

Moreover (1.10) *is also best possible in the sense that for $P_n(x) = x^n + \lambda x$ the expression on the left can be made arbitrarily close to the expression on the right-hand side by choosing λ positive and sufficiently large.*

COROLLARY. *Let $P_n(x)$ be an algebraic polynomial of degree n with nonnegative coefficients satisfying* (1.8) *with $a_n > 0$. Then for $\alpha \geqslant (\sqrt{5} - 1)/2$ we obtain*

$$\int_0^\infty P_n^2(x) x^\alpha e^{-x}\, dx \geqslant (a_n)^2 \Gamma(2n + \alpha + 1), \tag{1.11}$$

equality holding for $P_n(x) = x^n$ $(a_n = 1)$.

Proof of this Corollary is as follows. Repeated application of the inequality (1.9) gives at once

$$\frac{\int_0^\infty (P_n^k(x))^2 x^\alpha e^{-x}\, dx}{\int_0^\infty P_n^2(x) x^\alpha e^{-x}\, dx} \leqslant \frac{n^2(n-1)^2 \cdots (n-k+1)^2}{(2n+\alpha)(2n+\alpha-1)\cdots(2n+\alpha-2(k-1))(2n+\alpha-1-2(k-1))}. \tag{1.12}$$

Next, we note that

$$\int_0^\infty (P_n^{(n)}(x))^2 x^\alpha e^{-x}\, dx = (a_n)^2 (n!)^2 \Gamma(\alpha + 1). \tag{1.13}$$

Now, we substitute $k = n$ in (1.12) and use (1.13) which leads to (1.11).

2. Proof of Theorem 1. Let $P_n(x)$ be any algebraic polynomial of degree n with positive coefficients. Then we may write

$$P_n(x) = a_n x^n + P_{n-1}(x), \qquad P_{n-1}(x) = \sum_{k=0}^{n-1} a_k x^k, a_k \geqslant 0. \tag{2.1}$$

Using (2.1) and

$$\int_0^\infty x^k e^{-x}\, dx = \Gamma(k+1) \tag{2.2}$$

we obtain

(2.3)

$$\int_0^\infty (P_n'(x))^2 e^{-x} x^\alpha \, dx = \int_0^\infty (P_{n-1}'(x))^2 e^{-x} x^\alpha \, dx + a_n^2 n^2 \Gamma(2n + \alpha - 1) + 2na_n \int_0^\infty P_{n-1}'(x) x^{n+\alpha-1} e^{-x} \, dx,$$

and

(2.4)

$$\int_0^\infty (P_n(x))^2 e^{-x} x^\alpha \, dx = a_n^2 \Gamma(2n + \alpha + 1) + \int_0^\infty (P_{n-1}(x))^2 e^{-x} x^\alpha \, dx + 2a_n \int_0^\infty P_{n-1}(x) e^{-x} x^{n+\alpha} \, dx.$$

Next we set

$$b_n = \frac{n^2}{(2n + \alpha)(2n + \alpha - 1)} \tag{2.5}$$

and

$$\lambda_n = 2n \int_0^\infty P_{n-1}'(x) x^{n+\alpha-1} e^{-x} \, dx - 2b_n \int_0^\infty P_{n-1}(x) x^{n+\alpha} e^{-x} \, dx. \tag{2.6}$$

From (2.3)–(2.6) we may conclude that

$$\begin{aligned} &\int_0^\infty (P_n'(x))^2 e^{-x} x^\alpha \, dx - b_n \int_0^\infty (P_n(x))^2 e^{-x} x^\alpha \, dx \\ &\quad = \lambda_n a_n + \int_0^\infty (P_{n-1}'(x))^2 e^{-x} x^\alpha \, dx - b_n \int_0^\infty (P_{n-1}(x))^2 e^{-x} x^\alpha \, dx. \end{aligned} \tag{2.7}$$

First, we claim that for $\alpha \geqslant 0$

$$\lambda_n \leqslant 0, \qquad n = 1, 2, \ldots. \tag{2.8}$$

From (2.1), (2.2) and (2.6) we obtain

$$\lambda_n = \frac{2n}{(2n + \alpha)(2n + \alpha - 1)} \sum_{k=0}^{n-1} a_k \mu_{kn} \Gamma(k + n + \alpha - 1), \qquad a_k \geqslant 0, \tag{2.9}$$

where

$$\begin{aligned} \mu_{kn} &= (2n + \alpha)(2n + \alpha - 1)k - n(k + n + \alpha)(k + n + \alpha - 1) \\ &= (k - n)\{n(n - k) + (2\alpha - 1)n + \alpha(\alpha - 1)\}, \qquad 0 \leqslant k \leqslant n - 1. \end{aligned}$$

Clearly for $\alpha \geqslant 0$ we have

$$\mu_{kn} \leqslant -\alpha(2n + \alpha - 1) \leqslant 0, \qquad k = 0, 1, \ldots, n - 1. \tag{2.10}$$

Thus (2.8) follows easily from (2.9) and (2.10). We also note that

$$b_n \geqslant b_{n-1}, \qquad \alpha \geqslant \frac{\sqrt{5} - 1}{2}, \qquad n = 2, 3, \ldots, \tag{2.11}$$

and

$$b_n < b_{n-1}, \qquad 0 \leqslant \alpha \leqslant \frac{1}{2}, \qquad n = 2, 3, \ldots. \tag{2.12}$$

Using these ideas we can complete the proof of Theorem 1. For $\alpha \geqslant (\sqrt{5} - 1)/2$ we use (2.7), (2.8) and (2.11) and we obtain for $k = 2, 3, \ldots, n$,

$$\begin{aligned}\int_0^\infty (P_k'(x))^2 e^{-x} x^\alpha \, dx - b_k \int_0^\infty (P_k(x))^2 e^{-x} x^\alpha \, dx \\ \leqslant \int_0^\infty (P_{k-1}'(x))^2 e^{-x} x^\alpha \, dx - b_{k-1} \int_0^\infty (P_{k-1}(x))^2 e^{-x} x^\alpha \, dx.\end{aligned} \tag{2.13}$$

Adding all these equations we obtain

$$\begin{aligned}\int_0^\infty (P_n'(x))^2 e^{-x} x^\alpha \, dx - b_n \int_0^\infty (P_n(x))^2 e^{-x} x^\alpha \, dx \\ \leqslant \int_0^\infty (P_1'(x))^2 e^{-x} x^\alpha \, dx - b_1 \int_0^\infty (P_1(x))^2 e^{-x} x^\alpha \, dx.\end{aligned} \tag{2.14}$$

But a simple computation shows that if $P_1(x) = a_1 x + a_0$, $a_1 > 0$, $a_0 \geqslant 0$ then

$$\int_0^\infty (P_1'(x))^2 e^{-x} x^\alpha \, dx \leqslant b_1 \int_0^\infty (P_1(x))^2 e^{-x} x^\alpha \, dx, \tag{2.15}$$

equality holds if $P_1(x) = a_1 x$, $a_1 > 0$. From (2.14) and (2.15) we obtain (1.2) as desired for $\alpha \geqslant (\sqrt{5} - 1)/2$. For the proof of (1.3) for $0 \leqslant \alpha \leqslant \frac{1}{2}$ we use (2.7), (2.8) and (2.12). From these results it follows that

$$\begin{aligned}\int_0^\infty (P_k'(x))^2 e^{-x} x^\alpha \, dx - b_k \int_0^\infty (P_k(x))^2 e^{-x} x^\alpha \, dx \\ \leqslant \int_0^\infty (P_{k-1}'(x))^2 e^{-x} x^\alpha \, dx - b_{k-1} \int_0^\infty (P_{k-1}(x))^2 e^{-x} x^\alpha \, dx \\ + (b_{k-1} - b_k) \int_0^\infty (P_{k-1}(x))^2 e^{-x} x^\alpha \, dx.\end{aligned} \tag{2.16}$$

Also from (2.4) it follows that

$$\int_0^\infty P_{k-1}^2(x) x^\alpha e^{-x} \, dx \leqslant \int_0^\infty (P_k(x))^2 x^\alpha e^{-x} \, dx. \tag{2.17}$$

From (2.16) and (2.17) we obtain for $k = 2, 3, \ldots, n-1, n$,

$$\begin{aligned}\int_0^\infty (P_k'(x))^2 e^{-x} x^\alpha \, dx - b_k \int_0^\infty (P_k(x))^2 e^{-x} x^\alpha \, dx \\ \leqslant \int_0^\infty (P_{k-1}'(x))^2 e^{-x} x^\alpha \, dx - b_{k-1} \int_0^\infty (P_{k-1}(x))^2 e^{-x} x^\alpha \, dx \\ + (b_{k-1} - b_k) \int_0^\infty (P_n(x))^2 e^{-x} x^\alpha \, dx.\end{aligned} \tag{2.18}$$

We add all these equations for $k = 2, 3, \ldots, n-1, n$ and we obtain

$$\begin{aligned}\int_0^\infty (P_n'(x))^2 e^{-x} x^\alpha \, dx - b_n \int_0^\infty (P_n(x))^2 e^{-x} x^\alpha \, dx \\ \leqslant \int_0^\infty (P_1'(x))^2 e^{-x} x^\alpha \, dx - b_1 \int_0^\infty (P_1(x))^2 e^{-x} x^\alpha \, dx \\ + (b_1 - b_n) \int_0^\infty (P_n(x))^2 e^{-x} x^\alpha \, dx \\ \leqslant (b_1 - b_n) \int_0^\infty (P_n(x))^2 e^{-x} x^\alpha \, dx.\end{aligned} \tag{2.19}$$

But clearly (2.18) is equivalent to

$$\int_0^\infty (P_n'(x))^2 e^{-x} x^\alpha \, dx \leqslant b_1 \int_0^\infty (P_n(x))^2 e^{-x} x^\alpha \, dx. \tag{2.20}$$

From (2.19) and (2.5) follows (1.3).

In order to show that (1.3) is best possible we consider $f_0(x) = x^n + \lambda x$, $\lambda > 0$, $0 \leqslant \alpha \leqslant \frac{1}{2}$. A simple computation shows that

$$\frac{\int_0^\infty (f_0'(x))^2 e^{-x} x^\alpha \, dx}{\int_0^\infty (f_0(x))^2 e^{-x} x^\alpha \, dx} = \frac{1}{(\alpha + 2)(\alpha + 1)} - C_{n,\lambda},$$

where

$$C_{n,\lambda} = \frac{2\lambda\Gamma(n + \alpha)\left[\alpha(\alpha + 1) + n^2 - n(\alpha^2 + \alpha - 1)\right] + d_n}{(\alpha + 2)(\alpha + 1)\left[\lambda^2\Gamma(\alpha + 3) + 2\lambda\Gamma(n + \alpha + 2) + \Gamma(2n + \alpha + 1)\right]}$$

and

$$d_n = \Gamma(2n + \alpha - 1)\left[n^2(2 - 3\alpha - \alpha^2) + 2n(2\alpha - 1) + \alpha(\alpha - 1)\right].$$

Clearly $C_{n,\lambda} \to 0$ as $\lambda \to \infty$ from which our assertion follows.

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Department of Mathematics, University of Florida, Gainesville, Florida 32611