THE EQUIVARIANT EXTENSION THEOREM

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Abstract. A simplified proof of Jaworowski's equivariant extension theorem is given which enables one to generalize the domain to a class of $G$-spaces which include all (not necessarily compact) $G$-manifolds.

Let $G$ be a compact Lie group. In [2] Jaworowski proved the Extension Theorem: Let $X$ be a locally compact separable metric finite-dimensional $G$-space with a finite number of orbit types. Let $A$ be a closed invariant subspace and let $f: A \rightarrow Y$ be a $G$-map to a locally compact metric separable $G$-space $Y$. If $Y^H$ is an ANR (resp. AR) for every orbit type $(H)$ in $X - A$, then $f$ has a neighborhood $G$-extension (resp. a $G$-extension over $X$).

The purpose of this note is to give a simplified proof of the extension theorem which will enable us to extend Jaworowski's result to a class of $G$-spaces $X$ which include all (not necessarily compact) $G$-manifolds.

The following lemma is proved in Steenrod [4] for the case $Y$ is an AR; the case where $Y$ is an ANR requires only minor revisions.

Lemma 1. Let $p: E \rightarrow X$ be a locally trivial fibre bundle with fibre $Y$ an ANR (AR). Suppose $X$ is a normal Lindelöf space and $A \subseteq X$ a closed subspace. If $s: A \rightarrow p^{-1}(A)$ is a section over $A$, then $s$ can be extended to a section over a neighborhood of $A$ (over $X$).

Lemma 2 generalizes the well-known result on principal bundle maps, and may be proved in the same way, cf. Kosniowski [3].

Lemma 2. Let $X$ be a completely regular free $G$-space and $Y$ an arbitrary $G$-space. Then equivariant maps of $X$ into $Y$ are in bijective correspondence with cross-sections of the associated bundle $\varphi: X \times_G Y \rightarrow X/G$ with fibre $Y$ to the principal $G$-bundle $q: X \rightarrow X/G$.

Lemma 3. Let $A$ be a closed subspace of the topological space $X$. Let $Y$ be a metric space and $Z$ a compact space. Let $f: A \rightarrow Y$ and $g: X - A \rightarrow Z$ be continuous functions. If $\varphi: X \times Z \rightarrow Y$ is a continuous extension of $f \circ \pi_1, \pi_1: A \times Z \rightarrow A$, the projection onto the first coordinate, then the function $F: X \rightarrow Y, F|A = f, F|X - A = \varphi \circ (i, g), i: X - A \rightarrow X$ the inclusion, is a continuous extension of $f$. 

Recieved by the editors April 21, 1980 and, in revised form, June 6, 1980.

1980 Mathematics Subject Classification. Primary 55M15, 57S10, 57S15.

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0002-9939/81/0000-0431/$01.75

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Proof. Since $X - A$ is open, we need only prove continuity at points of $A$. If $a \in A$ and $\varepsilon > 0$, then, since $Z$ is compact, there is a neighborhood $U$ of $a$ in $X$ such that $d(\varphi_x, \varphi_a) < \varepsilon$, $x \in U$, $\varphi_x : Z \to Y$, and $\varphi_a(z) = \varphi(x, z)$. In particular, if $x \in U \cap (X - A)$, $d(\varphi(x, g(x)), \varphi(a, g(x))) < \varepsilon$. But $\varphi(a, g(x)) = f(a)$. We can also choose $U$ sufficiently small so that $x$ in $U \cap A$ implies $d(f(x), f(a)) < \varepsilon$. Hence for $x \in U$, $d(F(x), F(a)) < \varepsilon$. Thus $F$ is continuous.

Remark. If $X, Y$ and $Z$ are $G$-spaces, $A$ is invariant and $f, g$ and $\varphi$ are equivariant, then $F$ is equivariant.

The following lemma is proved, for example, in Bredon [1].

Lemma 4. For every $n$, there is a compact $n$-universal $G$-bundle. (Here $n$-universal means that it classifies $G$-bundles over paracompact spaces of covering dimension $< n$.)

Main Lemma. Let $X$ be a separable metric $G$-space and $A$ a closed invariant subspace such that $X - A$ is finite dimensional with a single orbit type $(H)$. Suppose $f : A \to Y$ is an equivariant map into a metrizable $G$-space. If $Y^H$ is an ANR (AR), then $f$ extends to an equivariant map on a neighborhood of $A$ (on $X$).

Proof. Let $X_H$ be the points of orbit type $(H)$ and let $X_H = X_{(H)} \cap X^H$. Then $[1]$, $X_{(H)} = G \times N(H) X_H = G/H \times K X_H$, $K = N(H)/H$. Hence a $G$-map of $X_{(H)}$ into $Y$ is completely determined by a $K$-map of $X_H$ into $Y^H$. Since $X - A \subset X_{(H')}$, to extend $f$ it is sufficient to extend the $K$-map $f^H : A^H \to Y^H$ to a neighborhood of $A^H$ in $X^H$ (to $X^H$). In fact, since $X^H$ is closed in $X$ and $G \times X \to X$ is closed, $G X^H$ is the quotient space of $G \times X^H$. Thus, if $F^H$ extends $f^H$ to a neighborhood $U$ of $A^H$ (to $X^H$), the map $F : G U \to Y (G X^H \to Y)$, $F(gx) = g F^H(x)$ is continuous and agrees with $f$ on $G U \cap A (G X^H \cap A)$ and hence $f$ extends to a neighborhood of $A$ (to $X = G X^H \cup A$).

Let $Z$ be a compact $n$-universal $K$-bundle, $n > \dim(X - A)/G$. Then, on the one hand, $X^H \times Z$ is a free $K$-space and, on the other, we have a $K$-bundle map of $X^H - A^H$ to $Z$. Now by Lemmas 1 and 2, $f^H \circ \pi_1 ; A \times Z \to Y^H$ extends to a $K$-map $\varphi$ of $U \times Z$, $U$ a neighborhood of $A^H$ (of $X^H \times Z$) into $Y$. By Lemma 3, this gives a $K$-extension of $f^H$ to $U$ (to $X^H$); and $f$ extends to a neighborhood of $A$ (to $X$).

Theorem A. Let $X$ be a separable metric $G$-space and $A$ a closed invariant subspace such that $X - A$ is finite dimensional with a finite number of orbit types. Suppose $f : A \to Y$ is an equivariant map into a metrizable $G$-space. If $Y^H$ is an ANR (AR) for each orbit type $(H)$ in $X - A$, then $f$ extends to an equivariant map on a neighborhood of $A$ (on $X$).

Proof. Partially order the orbit types $(H_1), (H_2), \ldots, (H_s)$ in $X - A$ so that if $(H_i) > (H_j)$ then $i > j$. Let $X_i = \bigcup_{j < i} X_{(H_j)}$. Then $X = A \cup X_i$. We inductively assume we have an equivariant extension of $f$ to $W_{i-1}$, $W_{i-1}$ an invariant neighborhood of $A$ in $A \cup X_{i-1}$ (to $A \cup X_{i-1}$). Then $W_{i-1} \cup X_{(H_i)} - W_{i-1}$ (resp. $A \cup X_i - A \cup X_{i-1}$) has a single orbit type and since closure$(X_{i-1})$ includes only points of
orbit type \( \leq (H_j), j < i - 1 \), \( W_{i-1} \) (resp. \( A \cup X_{i-1} \)) is closed in \( W_{i-1} \cup X_{(H_i)} \) (resp. \( A \cup X_i \)). By the Main Theorem, \( f \) extends to a neighborhood \( W_i \) of \( W_{i-1} \) in \( W_{i-1} \cup X_{(H_i)} \) (resp. \( A \cup X_i \)). Since

\[
W_{i-1} \cup X_{(H_i)} = A \cup X_i - (X_{i-1} - W_{i-1})
\]

is a neighborhood of \( A \) in \( A \cup X_i \), so is \( W_i \). Thus we have proved the inductive step and the result follows.

**Definition.** A \( G \)-space \( X \) has **locally finite orbit structure** if each orbit in \( X \) has a neighborhood with only finitely many orbit types.

Let \( A \) be an invariant subspace of \( X \). A \( G \)-space \( X \) has locally finite orbit structure mod \( A \) if each orbit \( G(x) \) in \( X \) has a neighborhood \( U_x \) with only finitely many orbit types in \( U_x - A \).

**Definition.** A separable metric space \( X \) has **locally finite dimension** if each \( x \in X \) has a finite-dimensional neighborhood.

**Definition.** A separable metric space \( X \) has **locally finite dimension** mod \( A \) if each \( x \in X \) has a neighborhood \( U_x \) such that \( U_x - A \) has finite dimension.


**Theorem B.** Let \( X \) be a separable metric \( G \)-space and \( A \) a closed invariant subspace such that \( X \) has locally finite dimension and locally finite orbit structure mod \( A \). Suppose \( f: A \to Y \) is an equivariant map into a metrizable \( G \)-space. If each \( Y^H \), \( (H) \) an orbit type of \( X - A \), is an ANR (AR), then \( f \) extends to an equivariant map on a neighborhood of \( A \) (on \( X \)).

**Proof.** Since \( X \) is normal, every neighborhood of an orbit contains a closed invariant neighborhood. Since \( X \) is Lindelöf, we can choose a countable number of these neighborhoods, \( V_1, V_2, \ldots \) such that \( \bigcup \text{Int} V_i = X \), and each \( V_i - A \) has only finitely many orbit types and is finite dimensional. Let \( X_i = \bigcup_{j < i} V_j \). Then \( X_i \) is closed, and \( \bigcup \text{Int} X_i = X \). By induction, it is sufficient to prove the result for \( A_i \subset X_i \), \( A_i = (A \cap X_i) \cup W^{i-1} \) (resp. \( (A \cap X_i) \cup X^{i-1} \)), when we assume \( f \) has been extended to \( A \cup W^{i-1} \) (resp. \( A \cup X^{i-1} \)), where \( W^{i-1} \) is a neighborhood of \( A \cap X^{i-1} \) in \( X^{i-1} \). But \( X_i - A_i \) has only finitely many orbit types and is finite dimensional. Hence, the inductive step follows from Theorem A, proving Theorem B.

**References**


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