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THE EQUIVARIANT EXTENSION THEOREM

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ABSTRACT. A simplified proof of Jaworowski's equivariant extension theorem is given which enables one to generalize the domain to a class of G-spaces which include all (not necessarily compact) G-manifolds.

Let G be a compact Lie group. In [2] Jaworowski proved the *Extension Theorem*: Let X be a locally compact separable metric finite-dimensonal G-space with a finite number of orbit types. Let A be a closed invariant subspace and let $f: A \to Y$ be a G-map to a locally compact metric separable G-space Y. If Y^H is an ANR (resp. AR) for every orbit type (H) in $X - A$, then f has a neighborhood G-extension (resp. a G-extension over X).

The purpose of this note is to give a simplified proof of the extension theorem which will enable us to extend Jaworowski's result to a class of G-spaces X which include all (not necessarily compact) G-manifolds.

The following lemma is proved in Steenrod [4] for the case Y is an AR; the case where Y is an ANR requires only minor revisions.

LEMMA 1. *Let $p: E \to X$ be a locally trivial fibre bundle with fibre Y an ANR (AR). Suppose X is a normal Lindelöf space and $A \subset X$ a closed subspace. If s: $A \to p^{-1}(A)$ is a section over A, then s can be extended to a section over a neighborhood of A (over X).*

Lemma 2 generalizes the well-known result on principal bundle maps, and may be proved in the same way, cf. Kosniowski [3].

LEMMA 2. *Let X be a completely regular free G-space and Y an arbitrary G-space. Then equivariant maps of X into Y are in bijective correspondence with cross-sections of the associated bundle $\varphi: X \times_G Y \to X/G$ with fibre Y to the principal G-bundle $q: X \to X/G$.*

LEMMA 3. *Let A be a closed subspace of the topological space X. Let Y be a metric space and Z a compact space. Let $f: A \to Y$ and $g: X - A \to Z$ be continuous functions. If $\varphi: X \times Z \to Y$ is a continuous extension of $f \circ \pi_1$, $\pi_1: A \times Z \to A$, the projection onto the first coordinate, then the function $F: X \to Y$, $F|A = f$, $F|X - A = \varphi \circ (i, g)$, $i: X - A \to X$ the inclusion, is a continuous extension of f.*

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PROOF. Since $X - A$ is open, we need only prove continuity at points of A. If $a \in A$ and $\varepsilon > 0$, then, since Z is compact, there is a neighborhood U of a in X such that $d(\varphi_x, \varphi_a) < \varepsilon$, $x \in U$, $\varphi_x\colon Z \to Y$, and $\varphi_x(z) = \varphi(x, z)$. In particular, if $x \in U \cap (X - A)$, $d(\varphi(x, g(x)), \varphi(a, g(x))) < \varepsilon$. But $\varphi(a, g(x)) = f(a)$. We can also choose U sufficiently small so that x in $U \cap A$ implies $d(f(x), f(a)) < \varepsilon$. Hence for $x \in U$, $d(F(x), F(a)) < \varepsilon$. Thus F is continuous.

REMARK. If X, Y and Z are G-spaces, A is invariant and f, g and φ are equivariant, then F is equivariant.

The following lemma is proved, for example, in Bredon [1].

LEMMA 4. *For every n, there is a compact n-universal G-bundle. (Here n-universal means that it classifies G-bundles over paracompact spaces of covering dimension $\leq n$.)*

MAIN LEMMA. *Let X be a separable metric G-space and A a closed invariant subspace such that $X - A$ is finite dimensional with a single orbit type (H). Suppose $f\colon A \to Y$ is an equivariant map into a metrizable G-space. If Y^H is an ANR (AR), then f extends to an equivariant map on a neighborhood of A (on X).*

PROOF. Let X_H be the points of orbit type (H) and let $X_H = X_{(H)} \cap X^H$. Then [1], $X_{(H)} = G \times_{N(H)} X_H = G/H \times_K X_H$, $K = N(H)/H$. Hence a G-map of $X_{(H)}$ into Y is completely determined by a K-map of X_H into Y^H. Since $X - A \subset X_{(H)}$, to extend f it is sufficient to extend the K-map $f^H\colon A^H \to Y^H$ to a neighborhood of A^H in X^H (to X^H). In fact, since X^H is closed in X and $G \times X \to X$ is closed, GX^H is the quotient space of $G \times X^H$. Thus, if F^H extends f^H to a neighborhood U of A^H (to X^H), the map $F\colon GU \to Y$ $(GX^H \to Y)$, $F(gx) = gF^H(x)$ is continuous and agrees with f on $GU \cap A$ $(GX^H \cap A)$ and hence f extends to a neighborhood of A (to $X = GX^H \cup A$).

Let Z be a compact n-universal K-bundle, $n \geq \dim(X - A)/G$. Then, on the one hand, $X^H \times Z$ is a free K-space and, on the other, we have a K-bundle map of $X^H - A^H$ to Z. Now by Lemmas 1 and 2, $f^H \circ \pi_1\colon A \times Z \to Y^H$ extends to a K-map φ of $U \times Z$, U a neighborhood of A^H (of $X^H \times Z$) into Y. By Lemma 3, this gives a K-extension of f^H to U (to X^H); and f extends to a neighborhood of A (to X).

THEOREM A. *Let X be a separable metric G-space and A a closed invariant subspace such that $X - A$ is finite dimensional with a finite number of orbit types. Suppose $f\colon A \to Y$ is an equivariant map into a metrizable G-space. If Y^H is an ANR (AR) for each orbit type (H) in $X - A$, then f extends to an equivariant map on a neighborhood of A (on X).*

PROOF. Partially order the orbit types $(H_1), (H_2), \ldots, (H_r)$ in $X - A$ so that if $(H_i) \geq (H_j)$ then $i \geq j$. Let $X_i = \bigcup_{j \leq i} X_{(H_j)}$. Then $X = A \cup X_r$. We inductively assume we have an equivariant extension of f to W_{i-1}, W_{i-1} an invariant neighborhood of A in $A \cup X_{i-1}$ (to $A \cup X_{i-1}$). Then $W_{i-1} \cup X_{(H_i)} - W_{i-1}$ (resp. $A \cup X_i - A \cup X_{i-1}$) has a single orbit type and since closure(X_{i-1}) includes only points of

orbit type $\leqslant (H_j), j \leqslant i-1$, W_{i-1} (resp. $A \cup X_{i-1}$) is closed in $W_{i-1} \cup X_{(H_i)}$ (resp. $A \cup X_i$). By the Main Theorem, f extends to a neighborhood W_i of W_{i-1} in $W_{i-1} \cup X_{(H_i)}$ (resp. $A \cup X_i$). Since

$$W_{i-1} \cup X_{(H_i)} = A \cup X_i - (X_{i-1} - W_{i-1})$$

is a neighborhood of A in $A \cup X_i$, so is W_i. Thus we have proved the inductive step and the result follows.

DEFINITION. A G-space X has *locally finite orbit structure* if each orbit in X has a neighborhood with only finitely many orbit types.

Let A be an invariant subspace of X. A G-space X has localy finite orbit structure mod A if each orbit $G(x)$ in X has a neighborhood U_x with only finitely many orbit types in $U_x - A$.

DEFINITION. A separable metric space X has *locally finite dimension* if each $x \in X$ has a finite-dimensional neighborhood.

Let A be a subspace of X. X has locally finite dimension mod A if each $x \in X$ has a neighborhood U_x such that $U_x - A$ has finite dimension.

REMARK. A G-manifold has locally finite orbit structure and locally finite dimension (cf. Chapters VI and VII of Borel's *Seminar on Transformation Groups*, Ann. of Math. Studies, no. 46, Princeton Univ. Press, Princeton, N. J., 1960).

THEOREM B. *Let X be a separable metric G-space and A a closed invariant subspace such that X has locally finite dimension and locally finite orbit structure* mod A. *Suppose f: $A \to Y$ is an equivariant map into a metrizable G-space. If each Y^H, (H) an orbit type of $X - A$, is an ANR (AR), then f extends to an equivariant map on a neighborhood of A (on X).*

PROOF. Since X is normal, every neighborhood of an orbit contains a closed invariant neighborhood. Since X is Lindelöf, we can choose a countable number of these neighborhoods, $V_1, V_2, \ldots$ such that $\bigcup \operatorname{Int} V_i = X$, and each $V_i - A$ has only finitely many orbit types and is finite dimensional. Let $X^i = \bigcup_{j<i} V_j$. Then X^i is closed, and $\bigcup \operatorname{Int} X^i = X$. By induction, it is sufficient to prove the result for $A^i \subset X^i$, $A^i = (A \cap X^i) \cup W^{i-1}$ (resp. $(A \cap X^i) \cup X^{i-1}$), when we assume f has been extended to $A \cup W^{i-1}$ (resp. $A \cup X^{i-1}$), where W^{i-1} is a neighborhood of $A \cap X^{i-1}$ in X^{i-1}. But $X^i - A^i$ has only finitely many orbit types and is finite dimensional. Hence, the inductive step follows from Theorem A, proving Theorem B.

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