CHAINS AND DISCRETE SETS IN ZERO-DIMENSIONAL
COMPACT SPACES

MURRAY BELL AND JOHN GINSBURG

Abstract. Let $X$ be a compact zero-dimensional space and let $B(X)$ denote the Boolean algebra of all clopen subsets of $X$. Let $\kappa$ be an infinite cardinal. It is shown that if $B(X)$ contains a chain of cardinality $\kappa$ then $X \times X$ contains a discrete subset of cardinality $\kappa$. This complements a recent result of J. Baumgartner and P. Komjath relating antichains in $B(X)$ to the $\pi$-weight of $X$.

1. Introduction. Let $X$ be a compact zero-dimensional space, and let $B(X)$ denote the Boolean algebra of clopen subsets of $X$. In this note we are interested in the relations between certain algebraic aspects of the Boolean algebra $B(X)$ and the topological properties of $X$. Specifically, we are concerned with chains and antichains in $B(X)$ and their connection with cardinal invariants of the space $X$.

Our set-theoretic and topological terminology and notation are standard. The cardinality of a set $S$ is denoted by $|S|$. All cardinal numbers considered in this paper are assumed to be infinite. Our basic reference for cardinality properties of topological spaces is [J].

For the reader's convenience we now recall several notions which will be useful in the sequel.

Recall that a topological space $D$ is said to be discrete if every point of $D$ is open in $D$. Thus if $D$ is a subspace of a space $X$, then $D$ is discrete if for every point $x$ of $D$ there is an open set $G_x$ in $X$ such that $G_x \cap D = \{x\}$. If $X$ is a topological space the spread of $X$, denoted by $s(X)$, is defined by

$$s(X) = \sup\{\kappa: \text{$X$ contains a discrete subset of cardinality } \kappa\}.$$ 

If $X$ is a space then the density character of $X$, denoted by $d(X)$, is defined by $d(X) = \min\{\kappa: \text{$X$ has a dense subset of cardinality } \kappa\}$. If $d(X) = \omega$ we say that $X$ is separable. The hereditary density character of $X$, denoted by $\overline{d}(X)$, is defined by $\overline{d}(X) = \sup\{d(S): S \subseteq X\}$. If $d(X) = \omega$ we say that $X$ is hereditarily separable.

A family $\Pi$ of nonempty open subsets of a space $X$ is called a $\pi$-base for $X$ if every nonempty open subset of $X$ contains a member of $\Pi$. The $\pi$-weight of $X$, denoted by $\pi(X)$, is defined by $\pi(X) = \min\{\kappa: \text{$X$ has a } \pi\text{-base of size } \kappa\}$. The hereditary $\pi$-weight of $X$, denoted by $\overline{\pi}(X)$, is defined to be

$$\overline{\pi}(X) = \sup\{\pi(S): S \subseteq X\}.$$
Clearly if $D$ is a discrete space of cardinality $\kappa$ then the density character of $D$ is $\kappa$. Therefore $s(X) < d(X)$. If $\Pi$ is a $\pi$-base for $X$ of cardinality $\kappa$ and if for each $P \in \Pi$ we choose a point $x_P$ in $P$, then the set $D = \{x_P: P \in \Pi\}$ is a dense subset of $X$ of cardinality at most $\kappa$. This shows that $d(X) < \pi(X)$ and hence that $d(X) < \pi(X)$.

Let $X$ be a space and let $(T, \prec)$ be a totally ordered set. Let $S = \{x_t: t \in T\}$ be a subset of $X$ indexed by $T$. $S$ is said to be totally separated by $T$ if $\{x_s: s \prec t\}$ is open in $S$ for every $t$ in $T$. We say that $S$ is a totally separated subspace of $X$.

If $B$ is a Boolean algebra then elements $x$ and $y$ of $B$ are said to be comparable if either $x \preceq y$ or $y \preceq x$; otherwise $x$ and $y$ are said to be incomparable. A subset $S$ of $B$ is called a chain if every two elements of $S$ are comparable. A subset $S$ of $B$ is called an antichain if no two elements of $S$ are comparable. Finally, a subset $S$ of $B$ is called a strong antichain if no element $x$ of $S$ is contained in the union of any finite subset of $S - \{x\}$; that is, $x \notin \bigvee F$ for any finite subset $F$ of $S - \{x\}$. Every strong antichain is an antichain but not conversely. The height of $B$ and the width of $B$ are defined respectively by $h(B) = \sup\{\kappa: B$ contains a chain of cardinality $\kappa\}$ and $w(B) = \sup\{\kappa: B$ contains an antichain of cardinality $\kappa\}$.

2. Chains in $B(X)$ and discrete subsets of $X \times X$. Throughout this section we assume that $X$ is a zero-dimensional compact space.

In [BK] it is shown that if all antichains of $B(X)$ have cardinality at most $\kappa$ then $X$ has a $\pi$-base of cardinality at most $\kappa$. That is, $\pi(X) < w(B(X))$. In [IN] it is pointed out that this result actually holds for hereditary $\pi$-weight: $\widehat{\pi}(X) < w(B(X))$. In particular, if all antichains of $B(X)$ are countable, then all subspaces of $X$ have countable $\pi$-weight and hence $X$ is hereditarily separable. A familiar example can be used to show that the above result cannot be improved to an equality: the Alexandroff-Urysohn “double-arrow” space (which is the same as the top and bottom of the lexicographically ordered unit square), is a compact zero-dimensional space which has an antichain of $c$ clopen sets. Furthermore, this space is hereditarily separable and first countable, and hence its hereditary $\pi$-weight is $\omega$. This example is also discussed in 9C of [E].

We will now establish a relation between chains of $B(X)$ and cardinal invariants of the space $X$.

2.1 Theorem. Let $\kappa$ be an infinite cardinal. If $B(X)$ contains a chain of cardinality $\kappa$ then $X$ contains a totally separated subspace of cardinality $\kappa$, and $X \times X$ contains a discrete subset of cardinality $\kappa$. In particular, $h(B(X)) < s(X \times X)$.

Proof. Let $\Lambda$ be a chain in $B(X)$ with $|\Lambda| = \kappa$. By deleting $\emptyset$ if necessary we may assume that every member of $\Lambda$ is nonempty. Let $A \in \Lambda$. We claim that $\bigcup\{B: B \in \Lambda$ and $B \subset A\} \subset A$. For $A$ is compact, being closed in $X$, and so, if the preceding union where equal to $A$ it follows that there would exist a finite number $B_1, B_2, \ldots, B_n$ of members of $\Lambda$ with $B_i \subset A$ for all $i = 1, 2, \ldots, n$ such that $B_1 \cup B_2 \cup \cdots \cup B_n = A$. But $\Lambda$ is a chain under inclusion so there is a largest member among $B_1, B_2, \ldots, B_n$, say $B_j$. Thus $B_j = A$. But this is impossible, since $B_j$ is a proper subset of $A$. This proves our claim. Thus for each $A$ in $\Lambda$ we may choose a point $x_A \in A - \bigcup\{B: B \in \Lambda$ and $B \subset A\}$. We note that the subspace
S = \{x_A: A \in \Lambda \} is totally separated by \Lambda under inclusion—in fact, for all A in \Lambda we have \{x_B: B \in \Lambda \text{ and } B \subseteq A \} = A \cap S, which implies that the set \{x_B: B \in \Lambda \text{ and } B \subseteq A \} is open in S. Thus S is a totally separated subspace of X having cardinality \kappa. This establishes the first part of the theorem.

Now, we can repeat the above argument applied to the chain \Lambda^* = \{X - A: A \in \Lambda \} (after deleting X from \Lambda if necessary to ensure that each member of \Lambda is proper). The result is a set of points \{y_A: A \in \Lambda \} such that, for all A, y_A \in X - A, and y_A \not\in X - B for any B in \Lambda with X - B \subseteq X - A. That is, y_A \not\in A and y_A \in B for all B in \Lambda such that A \subseteq B. We now claim that the set D = \{(x_A, y_A): A \in \Lambda \} is a discrete subset of X \times X, proving that X \times X has a discrete subset of cardinality \kappa as desired. This follows from the fact that A \times (X - A) is open in X \times X and A \times (X - A) \cap D = \{(x_A, y_A): A \in \Lambda \} for, if B \subseteq A then y_B \in A and so (x_B, y_B) \not\in A \times (X - A), while if A \subseteq B then x_B \not\in A and so again (x_B, y_B) \not\in A \times (X - A). This completes the proof.

An easy example shows that the inequality in our theorem cannot be improved to an equality: Let X be the one-point compactification of the discrete space of cardinality \kappa. Since all clopen subsets of X are either finite or cofinite, it follows that all chains in B(X) are countable. However X \times X, and indeed X itself, contains a discrete subset of cardinality \kappa.

It is worth noting that the existence of a discrete subset of X of cardinality \kappa does not follow from the existence of a chain in B(X) of cardinality \kappa. The double-arrow space serves as an example here also; it contains a chain of \aleph_0 clopen sets but contains no uncountable discrete subsets, since it is hereditarily separable.

As for discrete subsets of X we have the following simple result.

2.2 Theorem. Let X be a compact zero-dimensional space and let \kappa be an infinite cardinal. Then X contains a discrete subset of cardinality \kappa if and only if B(X) contains a strong antichain of cardinality \kappa.

Proof. Suppose X contains a discrete subset D with |D| = \kappa. Then for each x in D there is a clopen set A_x in X such that A_x \cap D = \{x\}. Clearly the collection \Lambda = \{A_x: x \in D \} is a strong antichain in B(X), since, for any x in D and any finite subset F of D - \{x\}, we have x \in A_x \cup \{A_y: y \in F \} and so A_x \not\subseteq \cup \{A_y: y \in F \}. Thus \Lambda is a strong antichain of cardinality \kappa.

Conversely, suppose \Lambda is a strong antichain in B(X) of cardinality \kappa. Let A be any member of \Lambda. The family of clopen sets \{A - B: B \in \Lambda - \{A\} \} has the finite intersection property, since A is not contained in the union of any finite number of members of \Lambda - \{A\}. Therefore, by compactness, \cap \{A - B: B \in \Lambda - \{A\} \} \neq \emptyset. Choose a point x_A in that intersection. Doing this for each A in \Lambda we construct a subset D = \{x_A: A \in \Lambda \} of X such that A \cap D = \{x_A\} for all A in \Lambda. Thus the set D is a discrete set of cardinality \kappa.

Combining 2.1 and 2.2 and using the obvious fact that X \times X contains a discrete set of cardinality \kappa whenever X does, we have the following corollary.

2.3 Corollary. If X \times X contains no discrete subsets of cardinality > \kappa then all chains and all strong antichains of B(X) have cardinality at most \kappa.
Remark. The conclusion of 2.2 has an obvious generalization to arbitrary compact spaces. One replaces the concept of a strong antichain of clopen sets by a set of pairs \( \{(F_i, G_i): i \in I\} \) where \( F_i \) is closed and \( G_i \) is open and \( F_i \subseteq G_i \) and such that for any \( i \) and any finite subset \( J \) of \( I - \{i\}, F_i \subseteq \bigcup \{G_j: j \in J\} \). There does not seem to be any natural generalization of 2.1 to arbitrary compact spaces.

References


Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2

Department of Mathematics, University of Winnipeg, Winnipeg, Manitoba, Canada R3B 2E9