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# ON IMMERSIONS OF $k$-MANIFOLDS IN $(2k-1)$-MANIFOLDS

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ABSTRACT. Let $f: M^k \to N^{2k-1}$. We prove that $f$ is homotopic to an immersion.

**1. Introduction.** Let $M, N$ be differentiable manifolds of dimension $k$ and $2k-1$, respectively, $k > 1$. It is well known that $M$ can be immersed in $R^{2k-1}$, so $M$ can also be immersed in $N$. We consider the problem of given a continuous map $f: M \to N$, whether there exists an immersion homotopic to $f$. In this note, we answer this problem affirmatively.

**2. Background.** In this section, we review some of the results of **[2]**. Let $N$ be a differentiable manifold. Let $T(N)$ denote the space of tangent vectors to $N$ and let $T_q(N)$ denote the space of all $q$-frames tangent to $N$. Let $(X_1, \ldots, X_q) \in T_q(N)$ and let $g = (a_{ij}) \in \mathrm{GL}(m)$, $m \leq q$. Define $(X_1, \ldots, X_q)g = (Y_1, \ldots, Y_q) \in T_q(N)$ by $Y_i = X_i$ if $i = m+1, \ldots, q$, and $Y_i = \sum_{j=1}^m a_{ji}X_j$, if $i = 1, \ldots, m$. This gives an action of $\mathrm{GL}(m)$ on $T_q(N)$ from the right.

Suppose $M$ is another manifold of dimension $m$ with $m < \dim N = n$. $T_m(M)$ is a bundle over $M$ with structural group $\mathrm{GL}(m)$. Let $\tilde{\mathfrak{B}}_q = \{p: B_q \to M\}$ be the associated bundle of $T_m(M)$ with $T_q(N)$ as fibre. By Hirsch's theory **[1]**, the existence of an immersion of $M$ in $N$ is equivalent to the existence of a cross section of $\tilde{\mathfrak{B}}_m$.

In **[2]**, another bundle $\mathfrak{B}_q = \{q: B_q \to M \times N\}$ with fibre $V_{n,q}$ is constructed so that the following diagram is commutative:

$$\begin{array}{ccc} & & B_q \\ & \overset{q}{\swarrow} & \\ M \times N & & \downarrow p \\ & \underset{p_1}{\searrow} & \\ & & M \end{array}$$

where $p_1$ is projection on the first factor. For any map $f: M \to N$, let $F: M \to M \times N$ be $F = \mathrm{id} \times f$. Let $\mathfrak{B}_q^f$ denote the bundle induced by $F$ from $\mathfrak{B}_q$ and $\mathfrak{B}_q^F$ denote the bundle $\mathfrak{B}_q | F(M)$. From **[2]** we have the following theorem.

THEOREM 2.1. *There is an immersion homotopic to $f: M \to N$ if and only if $\mathfrak{B}_m^f$ (or equivalently $\mathfrak{B}_m^F$) has a cross-section.*

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**3. Stiefel-Whitney classes.** Let $M$, $N$ be differentiable manifolds. Define the total Stiefel-Whitney class of $M$, $W(M) = 1 + W_1(M) + W_2(M) + \dots$, where $W_i(M)$ is the $i$th Stiefel-Whitney class of the tangent bundle of $M$. Then $\overline{W}(M)$, the total Stiefel-Whitney class of the normal bundle of $M$, is $W(M)^{-1}$. Let $f: M \to N$ be a continuous map. Define $W(f) = f^*(W(N)) \cdot W(M)^{-1}$. That is, if $f$ is an immersion, then $W(f)$ is the total Stiefel-Whitney class of the normal bundle of $M$ in $N$. Let $v(M) = 1 + v_1(M) + \dots$, be the total Wu class of $M$. If $M$ is a closed manifold, then the theorem of Wu [3] states that $W(M) = \mathrm{Sq}(v(M))$, where $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \dots$, is the total Steenrod square. Using [4], we can introduce right operations of the Steenrod algebra on $H^*(M)$ with respect to the normal bundle of $M$. These operations are defined by $(u)a \cdot U = \chi(a)(u \cdot U)$, where $U$ is the Thom class of the normal bundle and $\chi$ is the antiautomorphism of the Steenrod algebra. These operations satisfy the following proposition (see [4]).

PROPOSITION 3.1. (a) $(1)\chi(\mathrm{Sq}) = \overline{W}(M)$.

(b) *Let $M$ be a closed manifold. Then* $\langle (u)a \cdot v, [M]\rangle = \langle u \cdot a(v), [M]\rangle \in Z_2$, *where $[M]$ is the fundamental cycle of $M$ and $u, v \in H^*(M; Z_2)$.*

We will need the following theorem.

THEOREM 3.2. *Let $M$ be a closed manifold of* $\dim m$. *Then* $W_m(f) = f^*(v_m(N))$.

PROOF. We may assume that $M$ is connected. Then the theorem follows if we prove that $\langle W_m(f), [M]\rangle = \langle f^*(v_m(N)), [M]\rangle$.

$$\begin{aligned}\langle W_m(f), [M]\rangle &= \langle W(f), [M]\rangle = \langle W(M)^{-1} \cdot f^*(W(N)), [M]\rangle \\ &= \langle (1)_\chi(\mathrm{Sq}) \cdot f^*(\mathrm{Sq}(v(N))), [M]\rangle = \langle 1 \cdot_\chi(\mathrm{Sq})\mathrm{Sq}\, f^*(v(N)), [M]\rangle \\ &= \langle f^*(v(N)), [M]\rangle = \langle f^*(v_m(N)), [M]\rangle\end{aligned}$$

since $\chi(\mathrm{Sq})\mathrm{Sq} = 1$ is the defining property of $\chi$.

Let $M$ and $N$ be differentiable manifolds of dimensions $m$ and $2m$, respectively, where $m$ is even. Let $f: M \to N$ be an immersion. We have two cases. In case 1, we assume that $f^*(T(N))$ is orientable and there is no homotopy joining $f$ to $f$ and reversing the orientation of $f^*(T(N))$, and we define the $m$th normal class $w_m(f) \in H^m(M)$, where $H^m(M)$ denotes cohomology with local coefficients determined by the nonzero tangent vectors of $M$, as the Euler class of the normal bundle of $f$. In case 2, we assume that $f^*(T(N))$ is orientable and there is a homotopy joining $f$ to $f$ and reversing the orientation of $f^*(T(N))$, and we define an element $|w_m(f)| \in H^m_+(M) = H^m(M)/x \sim -x$, as a set, as the $m$th normal class of $f$. Note that $w_m(f)$ and $|w_m(f)|$ reduced mod 2 is $W_m(f)$. The following is proved in [2].

THEOREM 3.3. *Let $h$: $M \to N$ be a map such that $h^*(T(N))$ is orientable. Let $f$, $g$: $M \to N$ be two immersions homotopic to $h$. Then*

(1) *if any homotopy joining $h$ to $h$ preserves the orientation of $h^*(T(N))$, then $f$ and $g$ are regularly homotopic if and only if $w_m(f) = w_m(g)$;*

(2) *if there is a homotopy joining $h$ to $h$ which reverses the orientation of $h^*(T(N))$, then $f$ and $g$ are regularly homotopic if and only if $|w_m(f)| = |w_m(g)|$.*

*Then if $M$ is closed, the set of all $w_m(f)$ corresponding to all immersions $f$ homotopic to $h$ in* (1) *and the set of all $|w_m(f)|$ corresponding to all immersions $f$ homotopic to $h$ in* (2) *are in one-to-one correspondence with the set of even* [*odd*] *classes of $H^m(M)$ and $H^m_+(M)$, respectively, when $w_m(h) \equiv 0$ (2) [$w_m(h) \equiv 1$ (2)].*

COROLLARY 3.4. *If $M$ is closed, then the set of normal classes of immersions homotopic to $f$ is in one-to-one correspondence with the set of even* [*odd*] *classes of $H^m(M)$ or $H^m_+(M)$ when $f^*(v_m(N)) = 0$ [$f^*(v_m(N)) \neq 0$].*

We remark that if $|w_m(f)| = 0 \in H^m_+(M)$, then $w_m(f) = 0 \in H^m(M)$ is defined uniquely.

**4. The main theorem.** Let $f$: $M \to N$ be a continuous map, where $M$ and $N$ are manifolds of dimension $k$ and $2k-1$, respectively, $k > 1$. Let $N_1 = N \times R^1$, $i$: $N \to N_1$ is $i(x) = (x, 0)$. Define $f_1 = if$ and $F_1$: $M \to M \times N_1$ by $F_1 = (\mathrm{id} \times i)F$. We want to apply Theorem 2.1 and Corollary 3.4 to $f_1$. The following two lemmas are needed for the proof. Their proofs will be given in §5.

LEMMA 4.1. *If $\mathfrak{B}^{F_1}_{k+1}$ has a cross-section, then there is an immersion $g$: $M \to N$, $g$ homotopic to $f$.*

LEMMA 4.2. *If there is an immersion $g_1$: $M \to N_1$, $g_1$ homotopic to $f_1$, with $w_k(g_1) = 0$, then $\mathfrak{B}^{F_1}_{k+1}$ has a cross-section.*

Our main result is the following.

THEOREM 4.3. *Let $M$ and $N$ be differentiable manifolds of dimensions $k$ and $2k-1$, respectively, $k > 1$. Let $f$: $M \to N$. Then $f$ is homotopic to an immersion.*

PROOF.[3] By Lemmas 4.1 and 4.2, we need to find an immersion $g_1$: $M \to N_1$, homotopic to $f_1$, with $w_k(g_1) = 0$. We can assume $M$ is connected. If $M$ is not closed, any immersion $g_1$ must have $w_k(g_1) = 0$ and immersions exist. If $M$ is closed and $k$ is odd, and $f_1^*(T(N_1))$ is orientable, then for any immersion $g_1$ homotopic to $f_1$, $w_k(g_1)$ is an element of order 2 in $H^k(M) \approx Z$, so $w_k(g_1) = 0$. If $M$ and $N$ are both closed, $k$ is even, and $f_1^*(T(N_1))$ is orientable, then, by 3.4, we need only show $f_1^*(v_k(N_1)) = 0$. $v_k(N_1) = v_k(N) \otimes 1$, and $v_k(N) = 0$ for $N$ closed because

$$k > \frac{\dim N}{2} = \frac{2k-1}{2}.$$

Suppose $f_1^*(T(N_1))$ is not orientable, then the coefficient bundle $\mathfrak{B}$ determined by the nonzero normal vectors of an immersion $g_1$ homotopic to $f_1$ is not equivalent to the one determined by the nonzero tangent vectors of $M$. In this case, for any $k$ or any $M$, we have $H^k(M; \mathfrak{B}) = Z_2$. Let $\rho$: $\mathfrak{B} \to Z_2$, then $\rho_*$: $H^k(M; \mathfrak{B}) \to H^k(M; Z_2)$ is an isomorphism and $\rho_*(w_k(g_1)) = W_k(g_1)$. When $N$ is closed, $W_k(g_1) = f_1^*(v_k(N_1)) = 0$ as above. Finally, if $M$ is closed, $N$ is not closed and $k$ is even, we make the following construction to reduce it to the preceding case. We

[3] The referee points out that when $k$ is even, the fact that $w_k(f)$ is the only obstruction to a cross-section of $\mathfrak{B}^f_k$ follows from [5].

may assume $f(M) \cap \partial N = \varnothing$. $f(M)$ is a compact subset of $N - \partial N$ so there is a compact manifold $W$ such that $W \subset N - \partial N$ and $f(M) \subset W - \partial W$. Let $\overline{W}$ be the double of $W$. Let $f'$: $M \to W$, $i$: $W \to \overline{W}$. By the previous case, there is an immersion homotopic to $if'$: $M \to \overline{W}$. Hence, there is a monomorphism $\bar{f}_*$: $T(M) \to T(\overline{W})$ covering $if'$, and thus there is a monomorphism $f_*$: $T(M) \to T(W) \subset T(N)$ covering $f$: $M \to N$ and the result follows from Hirsch's theory **[1]**.

**5. Proofs of the lemmas.** Let $f$, $M$, $N$, $N_1$, $f_1$, $F_1$ be as in the first paragraph of §4. Let $p_2$: $M \times N_1 \to N_1$ and let $\mathfrak{B}' = \{q'\colon B' \to M \times N_1\}$ be the bundle induced by $p_2$ from the nonzero tangent vectors of $N_1$. Define $r$: $B_{k+1} \to B'$ by $r(X_1, \ldots, X_{k+1}) = X_{k+1}$. $r$ is a bundle map with fibre $V'_{2k,k} = \{(X_1, \ldots, X_k) \in V_{2k,k} | (X_1, \ldots, X_k, e) \in V_{2k,k+1}\}$, where $e$ is a fixed vector in $R^{2k}$. Furthermore,

$$\begin{array}{ccc} B_{k+1} & \xrightarrow{r} & B' \\ & \searrow_{q} \quad \swarrow_{q'} & \\ & M \times N_1 & \end{array}$$

commutes.

Proof of 4.1. Let $s$ be a section of $\mathfrak{B}^{F_1}_{k+1}$. Then $rs$ is a section of $\mathfrak{B}'_{F_1} = \mathfrak{B}'|F_1(M)$. Let $h$: $N_1 \to R^1$ and let $V$ be a nonzero tangent vector field on $R^1$. Then $hp_2$ induces a section $t$ of $\mathfrak{B}'$. The fibre of $\mathfrak{B}'_{F_1}$ is $R^{2k} - \{0\}$ and $k > 1$, so $t|F_1(M)$ and $rs$ are homotopic sections of $\mathfrak{B}'_{F_1}$. Thus by the covering homotopy property of $r$, there is a section $s_1$ of $\mathfrak{B}^{F_1}_{k+1}$ with $rs_1 = t|F_1(M)$. $s_1$ determines a monomorphism $f_*$: $T(M) \to T(N_1)$ such that for any $x \in M$, $f_*(T_x(M)) \subset T_{f(x)}(N_1)$ but $(h^* V)(f(x))$ does not belong to $f_*(T_x(M))$. Hence, there is a natural decomposition $T_{f(x)}(N_1) = T_{f(x)}(N) \oplus T_0(R^1)$ and $(h^* V)(f(x)) \in T_0(R^1)$. Thus $f_*$ followed by the projection of $T_{f(x)}(N_1)$ onto $T_{f(x)}(N)$, we obtain a monomorphism $\bar{f}_*$: $T(M) \to T(N)$. By Hirsch's theory [1], we see that there is an immersion $g$: $M \to N$ which is homotopic to $f$.

Proof of 4.2. Suppose $g_1$ is an immersion as in 4.2. Because $w_k(g) = 0$, there exists a normal vector field $V$ of $g_1$. $V$ together with the differential $dg_1$: $T(M) \to T(N_1)$ determines a section $s$ of $\mathfrak{B}^{G_1}_{k+1}$, where $G_1$: $M \to M \times N_1$ is defined by $g_1$. Define $h$: $F_1(M) \to G_1(M)$ by $h(x, f(x)) = (x, g(x))$. Then $sh$ is a mapping from $F_1(M)$ to $B_{k+1}$ with $qsh = h$. $h$ is homotopic to the identity, so by the covering homotopy property of $\mathfrak{B}_{k+1}$, there is a mapping $s_1$: $F_1(M) \to B_{k+1}$ such that $qs_1 =$ identity. Thus $s_1$ is a section of $\mathfrak{B}^{F_1}_{k+1}$ which proves the lemma.

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