ABELIAN $p$-GROUP ACTIONS ON HOMOLOGY SPHERES

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ABSTRACT. The Borel formula is extended to an identity covering actions of arbitrary Abelian $p$-groups. Specifically, suppose $G$ is an Abelian $p$-group which acts on a finite CW-complex $X$ which is a $\mathbb{Z}_p$-homology $n$-sphere. Each $X^H$ must be a $\mathbb{Z}_p$-homology $n(H)$-sphere and then

$$n - n(G) = \sum (n(K) - n(K/p)),$$

where the sum is over $A_0 = \{K|G/K is cyclic\}$ and the group $K/p$ is defined by

$$K/p = \{ g \in G | pg \in K \}.$$

This result is an immediate corollary of Theorem 2, whose converse Theorem 1, is also proven. Thus actions of Abelian $p$-groups on homology spheres resemble linear representations.

0. Introduction. When an elementary Abelian $p$-group $G$ acts on a mod-$p$ homology $n$-sphere $X$, each fixed point set $X^H$ is a $\mathbb{Z}_p$-homology sphere of dimension, say, $n(H)$. In [1, p. 175] it was first shown that

$$n - n(G) = \sum (n(H) - n(G))$$

where the sum runs over all corank 1 subgroups $H$. In [4], a converse result was established.

In this paper, we prove that a similar identity holds for the action of any Abelian $p$-group $G$. The converse is also proven.

Before we state the two main theorems we introduce some notation, which will be used at various points throughout the paper. If $K < G$ we set $K/p = \{ g \in G | pg \in K \}$, a subgroup of $G$. Also if $H < G$ we set

$$A_H = \{ K < G | K > H, G/K cyclic \},$$

so that, e.g. $A_0 = \{ all subgroups with cyclic quotient \}$. We can now state the main results.

THEOREM 1. Let $G$ be any finite Abelian $p$-group acting cellularly on a finite CW-complex $X$ such that for $H < G$, $H \neq 0$, $X^H$ is a $\mathbb{Z}_p$-homology $n(H)$-sphere. Assume that there exists $n > 0$ so that $\widetilde{H}_n(X; \mathbb{Z}_p) = 0$, $* \neq n$. For any subgroup $H$ of $G$ (including 0) assume the following identity holds for the $G/H$ action on $X^H$:

$$n(H) - n(G) = \sum (n(K) - n(K/p))$$

with the sum running over $A_H$. Assume that $n(K) - n(K/p)$ is even if $p$ is 2 and $|G/K| > 2$. Note that $n(0) = n$. Also assume $n - n(G)$ is even if $p$ is odd.

Then $H_n(X; \mathbb{Z}_p) = \mathbb{Z}_p + F$, with $F$ free over $\mathbb{Z}_pG$. "
Remark. Assuming \( n - n(G) \) even for \( p \) odd is a restriction only when \( G = \mathbb{Z}_p \).

**Theorem 2.** Let \( G \) be a finite Abelian \( p \)-group acting cellulary on a finite CW-complex \( X \) such that each \( X^H \) is a \( \mathbb{Z}_p \)-homology \( n(H) \)-sphere, \( H \neq 0 \). Suppose \( \tilde{H}_i(X; \mathbb{Z}_p) = 0, \ i \neq n, \) and \( H_n(X; \mathbb{Z}_p) = \mathbb{Z}_p + F, \ F \) free over \( \mathbb{Z}_p G \), for some integer \( n \). Then \( n - n(G) = \sum (n(K) - n(K/p)), \) sum over \( A_0 \).

We will indicate the proof of Theorem i in §i, \( i = 1, 2 \). Since the arguments here closely parallel those of [4], in general, only those necessary changes will be noted and where convenient the reader should see [4] for more complete details.

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1. In this section we prove Theorem 1, stated in the Introduction.

In order to prove Theorem 1, we construct an equivariant map \( \phi \) from \( X \) to an appropriate linear model \( S^n \). This map \( \phi \) will induce a \( \mathbb{Z}_p \)-homology isomorphism

\[
\phi_*^H: H_*(X^H; \mathbb{Z}_p) \rightarrow H_*(((S^n)^H; \mathbb{Z}_p) \quad \text{for all } H \neq 0,
\]

and an epimorphism when \( H = 0 \). Study of the mapping cone of \( \phi \) yields the theorem.

The linear model we need is provided by the following lemma.

**Lemma 1.1.** Let \( X, G \) and \( n \) be as in the statement of Theorem 1. Then there is a linear \( G \)-action on \( S^n \) such that for \( H < G \), if \( (S^n)^H = S^m \) then \( m = n(H) \) where \( X^H \) is a \( \mathbb{Z}_p \)-homology \( n(H) \)-sphere.

**Proof.** Let \( K \) be any subgroup of \( G \) such that \( G/K \) is cyclic and let \( \phi_K \) be a 1-dimensional complex (real, if \( |G/K| = 2 \)) irreducible representation of \( G/K \). For any representation \( \psi \), \( \psi^m \) denotes \( \psi + \psi + \cdots + \psi \) (\( m \)-times). If \( 1 \) denotes the trivial 1-dimensional real representation, consider the real representations,

\[
V = 1^{n(G)} + \sum (\phi_K) \frac{(n(K) - n(K/p))}{\alpha_K} \quad (p = 2)
\]

or

\[
V = 1^{n(G)} + \sum (\phi_K) \frac{(n(K) - n(K/p))}{2} \quad (p \neq 2)
\]

where \( \alpha_K = 1 \) if \( |G/K| = 2 \), \( \alpha_K = 2 \) if \( |G/K| > 2 \), and the sums are over \( A_0 \). In any case one can check that for any \( H < G \)

\[
V^H = 1^{n(G)} + \sum \phi_K \frac{(n(K) - n(K/p))}{\alpha_K} \quad (p = 2)
\]

or

\[
V^H = 1^{n(G)} + \sum \phi_K \frac{(n(K) - n(K/p))}{2} \quad (p \neq 2)
\]

with sums over \( A_H \). Thus \( \dim_\mathbb{R} V^K = n(K) \). The one-point compactification of \( V \) yields the required action on \( S^n \).
As in [4] we will assume that each sphere $(S^n)^H$ is 1-connected and that $X$ is a suspended $G$-space (this assumption does not affect the homological conclusion we wish to obtain).

The following two lemmas are proved essentially in [4, Lemmas 2 and 3].

**Lemma 1.2.** Let $X$ be a finite CW-complex with an action of a $p$-group $P$ such that all $X^H$ ($H < P$, $H \neq 0$) are $\mathbb{Z}_p$-acyclic. Suppose there is an integer $n > 0$ so that $H_i(X; \mathbb{Z}_p) = 0$, $i \neq n$. Then $H_n(X; \mathbb{Z}_p)$ is a free $\mathbb{Z}_p[P]$-module.

**Lemma 1.3.** Given $G, X, S^n$ as above, suppose $\phi : X \to S^n$ is a $G$-map which induces a $\mathbb{Z}_p$-homology isomorphism $\phi^H$ for $H \neq 0$ and an epimorphism for $H = 0$ where $\phi^H : X^H \to (S^n)^H$. Then $H_n(X; \mathbb{Z}_p) = \mathbb{Z}_p + F$, $F$ a free $\mathbb{Z}_pG$-module.

We must now show that the map $\phi$ exists. By Lemma 4 of [4] there is a map $\phi^G : X^G \to (S^n)^G$ which induces a $\mathbb{Z}_p$-homology isomorphism (by standard obstruction theory and the fact that $X^G$ is a co-$H$-space) (see [4] for a proof).

Obstructions to extending $\phi^G$ to an equivariant map $\phi : X \to S^n$ lie in the groups $H^{k+1}_G(X, X^G; \mathbb{Z}_k(S^n))$, the equivariant classical cohomology groups defined by Bredon in [3]. Using the arguments of [4] one can show that these groups all consist of torsion prime to $p$. Since $X$ (and all its skeleta) are co-$H$-spaces, a simple procedure using cogroup addition allows one to circumvent any possible obstructions which might occur. (See proof of Lemma 4 of [4].) As in [4], one needs to establish a vanishing result similar to the one given in [4, p. 284], namely that $H_i(X^H, (X^G)^H; \mathbb{Z}_p) = 0$ for $i > n(H)$, where $X_1$ is the subcomplex of points in $X$ whose isotropy subgroup has order $p$ or greater. One proves the stronger statement that $H_i(X^H, \cup_S X^M; \mathbb{Z}_p) = 0$ for $i > n(H)$, where $S$ is any collection of subgroups of $G$ which contain $H$, by Mayer-Vietoris and induction on order of $H$ and cardinality of $S$ (see [5, p. 587]).

The fact that $\phi$, once obtained, has the correct properties is Smith theory and is demonstrated in [4, Lemma 9].

2. We now wish to indicate the proof of Theorem 2, which follows the argument given for Proposition 1 of [5].

Let $k$ be the cellular dimension of $X$, construct $Y_{k-1}$, a $k$-dimensional $k$-1-connected $G$-CW-complex such that

$$\text{Ext}^s_{\mathbb{Z}_pG}(k-n)(H_n(X; \mathbb{Z}_p), \mathbb{Z}_p) = \text{Ext}^s_{\mathbb{Z}_pG}(H_k(Y_{k-1}; \mathbb{Z}_p), \mathbb{Z}_p)$$

for all $s > 2$.

Another finite $G$-CW complex $\tilde{Y}_{k-1}$ is defined which satisfies all the hypotheses of Theorem 1 for the integer $m = \Sigma(n(K) - n(K/p)) + n(G)$ (sum over $A_0$). It can be shown (e.g. [5, p. 588]) that

$$\text{Ext}^s_{\mathbb{Z}_pG}(k-n)(H_n(X; \mathbb{Z}_p), \mathbb{Z}_p) = \text{Ext}^s_{\mathbb{Z}_pG}(k-m)(\tilde{Y}_{k-1}; \mathbb{Z}_p), \mathbb{Z}_p).$$

It follows that $m = n$ because unless $G$ is cyclic, $G$ has nonperiodic cohomology. When $G$ is cyclic, the conclusion of Theorem 2 is trivial. This completes the discussion of the arguments establishing Theorems 1 and 2.
REFERENCES


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