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ABELIAN p-GROUP ACTIONS ON HOMOLOGY SPHERES

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ABSTRACT. The Borel formula is extended to an identity covering actions of arbitrary Abelian p-groups. Specifically, suppose G is an Abelian p-group which acts on a finite CW-complex X which is a Z_p-homology n-sphere. Each X^H must be a Z_p-homology $n(H)$-sphere and then

$$n - n(G) = \sum (n(K) - n(K/p))$$

where the sum is over $A_0 = \{K | G/K \text{ is cyclic}\}$ and the group K/p is defined by

$$K/p = \{g \in G | pg \in K\}.$$

This result is an immediate corollary of Theorem 2, whose converse Theorem 1, is also proven. Thus actions of Abelian p-groups on homology spheres resemble linear representations.

0. Introduction. When an elementary Abelian p-group G acts on a mod-p homology n-sphere X, each fixed point set X^H is a Z_p-homology sphere of dimension, say, $n(H)$. In [**1**, p. 175] it was first shown that

$$n - n(G) = \sum (n(H) - n(G))$$

where the sum runs over all corank 1 subgroups H. In [**4**], a converse result was established.

In this paper, we prove that a similar identity holds for the action of any Abelian p-group G. The converse is also proven.

Before we state the two main theorems we introduce some notation, which will be used at various points throughout the paper. If $K < G$ we set $K/p = \{g \in G | pg \in K\}$, a subgroup of G. Also if $H < G$ we set

$$A_H = \{K < G | K > H, G/K \text{ cyclic}\},$$

so that, e.g. $A_0 = \{\text{all subgroups with cyclic quotient}\}$. We can now state the main results.

THEOREM 1. *Let G be any finite Abelian p-group acting cellularly on a finite CW-complex X such that for $H \leqslant G$, $H \neq 0$, X^H is a Z_p-homology $n(H)$-sphere. Assume that there exists $n > 0$ so that $\tilde{H}_*(X; Z_p) = 0$, $* \neq n$. For any subgroup H of G (including 0) assume the following identity holds for the G/H action on X^H:*

$$n(H) - n(G) = \sum (n(K) - n(K/p))$$

with the sum running over A_H. Assume that $n(K) - n(K/p)$ is even if p is 2 and $|G/K| > 2$. Note that $n(0) = n$. Also assume $n - n(G)$ is even if p is odd.

Then $H_n(X; Z_p) = Z_p + F$, with F free over Z_pG.

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REMARK. Assuming $n - n(G)$ even for p odd is a restriction only when $G = Z_p s$.

THEOREM 2. *Let G be a finite Abelian p-group acting cellulary on a finite CW-complex X such that each X^H is a Z_p-homology $n(H)$-sphere, $H \neq 0$. Suppose $\tilde{H}_i(X; Z_p) = 0$, $i \neq n$, and $H_n(X; Z_p) = Z_p + F$, F free over Z_pG, for some integer n. Then $n - n(G) = \Sigma(n(K) - n(K/p))$, sum over A_0.*

We will indicate the proof of Theorem i in §i, $i = 1, 2$. Since the arguments here closely parallel those of [4], in general, only those necessary changes will be noted and where convenient the reader should see [4] for more complete details.

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1. In this section we prove Theorem 1, stated in the Introduction.

In order to prove Theorem 1, we construct an equivariant map ϕ from X to an appropriate linear model S^n. This map ϕ will induce a Z_p-homology isomorphism

$$\phi_*^H: H_*\left(X^H; Z_p\right) \to H_*\left((S^n)^H; Z_p\right) \quad \text{for all } H \neq 0,$$

and an epimorphism when $H = 0$. Study of the mapping cone of ϕ yields the theorem.

The linear model we need is provided by the following lemma.

LEMMA 1.1. *Let X, G and n be as in the statement of Theorem 1. Then there is a linear G-action on S^n such that for $H < G$, if $(S^n)^H = S^m$ then $m = n(H)$ where X^H is a Z_p-homology $n(H)$-sphere.*

PROOF. Let K be any subgroup of G such that G/K is cyclic and let ϕ_K be a 1-dimensional complex (real, if $|G/K| = 2$) irreducible representation of G/K. For any representation ψ, ψ^m denotes $\psi + \psi + \cdots + \psi$ (m-times). If 1 denotes the trivial 1-dimensional real representation, consider the real representations,

$$V = 1^{n(G)} + \sum (\phi_K)^{\frac{(n(K) - n(K/p))}{\alpha_K}} \qquad (p = 2)$$

or

$$V = 1^{n(G)} + \sum (\phi_K)^{\frac{(n(K) - n(K/p))}{2}} \qquad (p \neq 2)$$

where $\alpha_K = 1$ if $|G/K| = 2$, $\alpha_K = 2$ if $|G/K| > 2$, and the sums are over A_0. In any case one can check that for any $H < G$

$$V^H = 1^{n(G)} + \sum \phi_K^{\frac{(n(K) - n(K/p))}{\alpha_K}} \qquad (p = 2)$$

or

$$V^H = 1^{n(G)} + \sum \phi_K^{\frac{(n(K) - n(K/p))}{2}} \qquad (p \neq 2)$$

with sums over A_H. Thus $\dim_{\mathbf{R}} V^K = n(K)$. The one-point compactification of V yields the required action on S^n.

As in [4] we will assume that each sphere $(S^n)^H$ is 1-connected and that X is a suspended G-space (this assumption does not affect the homological conclusion we wish to obtain).

The following two lemmas are proved essentially in [4, Lemmas 2 and 3].

LEMMA 1.2. *Let X be a finite CW-complex with an action of a p-group P such that all X^H ($H \leqslant P$, $H \neq 0$) are Z_p-acyclic. Suppose there is an integer $n > 0$ so that $\tilde{H}_i(X; Z_p) = 0$, $i \neq n$. Then $H_n(X; Z_p)$ is a free $Z_p[P]$-module.*

LEMMA 1.3. *Given G, X, S^n as above, suppose ϕ: $X \to S^n$ is a G-map which induces a Z_p-homology isomorphism ϕ^H for $H \neq 0$ and an epimorphism for $H = 0$ where ϕ^H: $X^H \to (S^n)^H$. Then $H_n(X; Z_p) = Z_p + F$, F a free Z_pG-module.*

We must now show that the map ϕ exists. By Lemma 4 of [4] there is a map ϕ^G: $X^G \to (S^n)^G$ which induces a Z_p-homology isomorphism (by standard obstruction theory and the fact that X^G is a co-H-space) (see [4] for a proof).

Obstructions to extending ϕ^G to an equivariant map ϕ: $X \to S^n$ lie in the groups $H_G^{k+1}(X, X^G; \tilde{\omega}_k(S^n))$, the equivariant classical cohomology groups defined by Bredon in [3]. Using the arguments of [4] one can show that these groups all consist of torsion prime to p. Since X (and all its skeleta) are co-H-spaces, a simple procedure using cogroup addition allows one to circumvent any possible obstructions which might occur. (See proof of Lemma 4 of [4].) As in [4], one needs to establish a vanishing result similar to the one given in [4, p. 284], namely that $H_i(X^H, (X_1)^H; Z_p) = 0$ for $i > n(H)$, where X_1 is the subcomplex of points in X whose isotropy subgroup has order p or greater. One proves the stronger statement that $H_i(X^H, \bigcup_S X^M; Z_p) = 0$ for $i > n(H)$, where S is any collection of subgroups of G which contain H, by Mayer-Vietoris and induction on order of H and cardinality of S (see [5, p. 587]).

The fact that ϕ, once obtained, has the correct properties is Smith theory and is demonstrated in [4, Lemma 9].

2. We now wish to indicate the proof of Theorem 2, which follows the argument given for Proposition 1 of [5].

Let k be the cellular dimension of X, construct Y_{k-1}, a k-dimensional k-1-connected G-CW-complex such that

$$\mathrm{Ext}_{Z_pG}^{s+(k-n)}(H_n(X; Z_p), Z_p) = \mathrm{Ext}_{Z_pG}^{s}(H_k(Y_{k-1}; Z_p), Z_p)$$

for all $s \geqslant 2$.

Another finite G-CW complex $\hat{Y}_{k-1}$ is defined which satisfies all the hypotheses of Theorem 1 for the integer $m = \Sigma(n(K) - n(K/p)) + n(G)$ (sum over A_0). It can be shown (e.g. [5, p. 588]) that

$$\mathrm{Ext}_{Z_pG}^{s+(k-n)}(H_n(X; Z_p), Z_p) = \mathrm{Ext}_{Z_pG}^{s+(k-m)}((\hat{Y}_{k-1}; Z_p), Z_p).$$

It follows that $m = n$ because unless G is cyclic, G has nonperiodic cohomology. When G is cyclic, the conclusion of Theorem 2 is trivial. This completes the discussion of the arguments establishing Theorems 1 and 2.

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