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ON ATRIODIC TREE-LIKE CONTINUA[1]

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ABSTRACT. D. P. Bellamy has recently shown that atriodic tree-like continua do not have the fixed point property for homeomorphisms. J. B. Fugate and T. B. McLean showed that hereditarily indecomposable tree-like continua have the fixed point property for pointwise periodic homeomorphisms. In this paper the latter result is extended to the case of atriodic tree-like continua. In the course of the proof it is shown that the property of being an atriodic tree-like continuum is a Whitney property. In particular, it is shown that the hyperspace of an atriodic tree-like continuum is at most 2-dimensional.

1. Introduction. A *continuum* is a compact, connected, metric space. A *tree* is a finite, connected, simply connected, one-dimensional polyhedron. A continuum is *tree-like* if it admits finite open covers of arbitrarily small mesh whose nerves are trees. A continuum X is said to be a *triod* (resp. n-od) if there exists a subcontinuum M of X such that $X \setminus M$ has at least three (resp. at least n) components. We say X is *atriodic* if X contains no triod. A continuum is hereditarily indecomposable if and only if it contains no 2-od.

If X is a continuum we let $C(X)$ denote the hyperspace of subcontinua of X with the Hausdorff metric. A *Whitney map* for X is a mapping μ: $C(X) \to [0, \infty)$ such that $\mu(\{x\}) = 0$ for each $x \in X$ and $\mu(A) < \mu(B)$ for each $A, B \in C(X)$ with $A \subsetneq B$. A *Whitney level* for X is a set $\mu^{-1}(t)$ where $0 < t < \mu(X)$. Whitney levels are continua in $C(X)$ (see [**9**, p. 400]). The existence of Whitney maps for X is well known (see [**9**]).

2. Whitney property. A property P of continua is said to be a *Whitney property* if whenever X is a continuum with property P then every Whitney level of X also has property P. Krasinkiewicz in [**6**] and [**7**] proved that being an arc-like continuum, being a proper circle-like continuum or being an hereditarily indecomposable tree-like continuum is a Whitney property. The main purpose of this section is to show that being an atriodic tree-like continuum is also a Whitney property. This provides a converse to a result of Nadler [**8**, 3.5] who has shown that if X is a continuum whose Whitney levels are tree-like then X is atriodic and tree-like.

A continuum X is said to have the *covering property* (see [**9**]) if for each Whitney level $\mu^{-1}(t)$ of X and each subcontinuum Λ of $\mu^{-1}(t)$, $\bigcup \Lambda = X$ implies $\Lambda = \mu^{-1}(t)$.

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THEOREM 2.1 (SEE [9, p. 485]). *A continuum X has the covering property if and only if each Whitney level of X is irreducible.*

THEOREM 2.2 [4, 5.6]. *Atriodic tree-like continua have the covering property.*

LEMMA 2.3. *Let X be an atriodic tree-like continuum. Let $p \in X$ and let μ be a Whitney map for X. If Λ is a subcontinuum of $\mu^{-1}(t)$ for some Whitney level then $K = \Lambda \cap \{A \in \mu^{-1}(t) \mid p \in A\}$ is an arc or a point or K is empty.*

PROOF. We suppose $p \in \bigcup \Lambda$. Since $\bigcup \Lambda$ is a continuum, it has the covering property by Theorem 2.2. Now $\mu|_{C(\bigcup \Lambda)}$ is a Whitney map for $\bigcup \Lambda$ and hence Λ is a Whitney level of $\bigcup \Lambda$. By [9, p. 405] $\{A \in \Lambda \mid p \in A\} = K$ is an arcwise connected continuum.

Now $\bigcup K$ is an atriodic tree-like continuum. By Theorem 2.2 $\bigcup K$ has the covering property. Hence, K is a Whitney level of $\bigcup K$. By Theorem 2.1 K is irreducible.

Sorgenfrey proved in [13, Theorem 1.8] that if T is the union of three continua which have a point in common and such that no one of them is a subset of the union of the other two then T contains a triod.

THEOREM 2.4. *The property of being an atriodic tree-like continuum is a Whitney property.*

PROOF. Let X be an atriodic tree-like continuum and let $\mu^{-1}(t)$ be a Whitney level of X. If Λ is a subcontinuum of $\mu^{-1}(t)$ then as in the proof of Lemma 2.3 Λ is a Whitney level of the tree-like continuum $\bigcup \Lambda$. By [11, Theorem 5] the first Čech cohomology group of Λ is trivial.

It follows that $\dim \mu^{-1}(t) = 1$. For if Y is a continuum with $\dim Y \geqslant 2$ then there exists an essential map f of Y onto B, the closed unit disk in the plane [10, p. 127]. Then $f|_{f^{-1}(S^1)}$ is essential where S^1 is the boundary of B and, hence, f is essential on some subcontinuum Z of $f^{-1}(S^1)$. i.e. $H^1(Z) \neq 0$.

Let $K = \{(A, x) \mid x \in A \in \mu^{-1}(t)\} \subset \mu^{-1}(t) \times X$. Then K is a continuum and $\dim K \leqslant 2$. Let $\pi_1: K \to \mu^{-1}(t)$ and $\pi_2: K \to X$ be the coordinate projections. The point inverses under π_1 are tree-like continua and the point inverses under π_2 are arcs or points by Lemma 2.3. In particular, the point inverses of π_1 and π_2 have trivial shape. By a theorem of Sher [12] $\mu^{-1}(t)$ and X have the same shape since π_1 and π_2 are cell-like mappings between finite dimensional spaces. By the theorem of Case and Chamberlin [2] $\mu^{-1}(t)$ is tree-like.

By Theorem 2.2 X has the covering property hereditarily. Therefore, by [9, p. 510] $\mu^{-1}(t)$ is hereditarily irreducible and, hence, $\mu^{-1}(t)$ is atriodic. This completes the proof of the theorem.

COROLLARY 2.5. *If X is an atriodic tree-like continuum, then $C(X)$ is 2-dimensional.*

3. The fixed point theorem. In [3] Fugate and McLean proved the following two results.

THEOREM 3.1 [3, 1.5]. *Tree-like continua have the fixed point property for periodic homeomorphisms.*

THEOREM 3.2 [3, 3.3]. *Hereditarily indecomposable tree-like continua have the fixed point property for pointwise periodic homeomorphisms.*

In this section we extend Theorem 3.2 to the case of atriodic tree-like continua. In our argument we use Theorem 2.4 and follow the argument given in [3]. First we prove the following lemma.

LEMMA 3.3. *If M is an atriodic, hereditarily unicoherent continuum and if h: $M \to M$ is a pointwise periodic homeomorphism, then the induced homeomorphism $\bar{h}$: $C(M) \to C(M)$ which is defined by $\bar{h}(Y) = h(Y)$ for each $Y \in C(M)$ is pointwise periodic. Moreover, if $x \in A \in C(M)$ and $h^n(x) = x$, then $\bar{h}^{2n}(A) = A$.*

PROOF. Let $x \in A \in C(M)$ and suppose $h^n(x) = x$. If $y \in A$ then there is a unique continuum B_y in M which is irreducible from x to y since M is hereditarily unicoherent. Then $B_y \subset A$ and $A = \bigcup\{B_y \mid y \in A\}$. Let $y \in A \setminus \{x\}$ and let $B = B_y$. It suffices to show that $h^{2n}(B) = B$. Suppose $h^{2n}(B) \neq B$. If $B \subsetneq h^n(B)$ let $z \in h^n(B) \setminus B$. Then $h^{in}(z) \in h^{(i+1)n}(B) \setminus h^{in}(B)$ for each i. Since $h^{in}(B) \subset h^{(i+1)n}(B)$ for each positive integer i this would imply z has infinite order under h and so would contradict the pointwise periodicity of h. Thus, $B \not\subset h^n(B)$. Similarly, $h^n(B) \not\subset B$. Notice that $h^n(y) \notin B$ since $B \cap h^n(B)$ is a proper subcontinuum of $h^n(B)$ and $h^n(B)$ is irreducible between $h^n(x) = x$ and $h^n(y)$. Similarly, $h^n(y) \notin h^{2n}(B)$, $h^{2n}(y) \notin B \cup h^n(B)$ and $y \notin h^n(B) \cup h^{2n}(B)$. By Sorgenfrey's theorem [13, 1.8] M contains a triod. This is a contradiction. Thus, we have proved $B = h^{2n}(B)$ and, hence, $A = \bar{h}^{2n}(A)$.

THEOREM 3.4. *Suppose M is an atriodic tree-like continuum and h: $M \to M$ is a pointwise periodic homeomorphism. Then h has a fixed point.*

PROOF. Suppose h does not have a fixed point. We may suppose M is minimal with respect to being mapped into itself. Hence, M is not a point and if Y is a proper subcontinuum of M, $h(Y) \not\subset Y$.

Let μ be a Whitney map for M. We may suppose $\mu(M) = 1$. Let $\bar{h}$: $C(M) \to C(M)$ be the map induced by h.

For $x \in M$ let $O(x) = \min_{n>0}(h^n(x) = x)$. Let $J_i = \{x \in M \mid O(x) \leqslant i\}$. Then J_i is closed and $M = J_1 \cup J_2 \cup \ldots$. By the Baire Category Theorem there exists n such that J_n has nonvoid interior in M. From the above it follows that there exists s with $0 < s < 1$ such that if $K \in \mu^{-1}([s, 1])$ then $K \cap J_n \neq \varnothing$.

Define σ: $C(M) \to [0, 1]$ by

$$\sigma(A) = \max\left\{ \mu\left(\bar{h}^i(A)\right) \mid 1 \leqslant i \leqslant 2n! \right\}.$$

Then σ is clearly a Whitney map for M such that if $\mu(A) \geqslant s$ then $\sigma(\bar{h}(A)) = \sigma(A)$. Since $\mu^{-1}(1) = \sigma^{-1}(1) = \{M\}$ there exists $0 < t < 1$ such that $\sigma(A) \geqslant t$ implies $\mu(A) \geqslant s$.

By Theorem 2.4 $\sigma^{-1}(t)$ is a tree-like continuum in $C(M)$. The restriction of $\bar{h}$ to $\sigma^{-1}(t)$ is a periodic homeomorphism of $\sigma^{-1}(t)$ (of period $\leqslant 2n!$). By Theorem 3.1 $\bar{h}(A) = A$ for some $A \in \sigma^{-1}(t)$. This contradicts the assumption at the beginning of the proof that $\bar{h}(Y) \not\subset Y$ for each proper subcontinuum Y of M.

Question. Is Theorem 3.4 true for tree-like continua which do not contain n-ods for arbitrarily large n?

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