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A NOTE ON h-PROXIMITIES

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ABSTRACT. It is shown that regularity and local bicompactness of the space (X, c) is a necessary condition for the existence of a largest h-proximity on (X, c). That these conditions are sufficient was shown earlier by the authors.

In [1] the authors showed that a topological space admits a largest h-proximity (R-proximity in the sense of Harris) if it is regular and locally bicompact. In this note we shall show that these conditions are necessary also.

The first three theorems of this article are from [1]. The notation used here is the same as in [1]. The reader is therefore kindly referred to it.

Let X be a nonempty set. A family $\mathcal{G} \subset \mathcal{P}(X)$ is a grill iff it is a union of ultrafilters on X. $\mathcal{G}$ is called a nonprincipal grill or an n.p. grill if it contains no singleton sets. We shall denote the family of grills on X by Γ.

A proximity π on X is a relation on $\mathcal{P}(X)$ satisfying the following properties:

(1) $\pi = \pi^{-1}$.

(2) $\pi(A)$ is a grill X for all subsets A of X.

(3) $\pi(A) \supset \bigcup [\mathcal{U}: A \in \mathcal{U}] \forall A \subset X$ where $\mathcal{U}$ represents an ultrafilter on X.

By R we shall mean the family relations on X. Define the function h: $R \times \Gamma \to \mathcal{P}(X)$ as follows: $h(\Psi, \mathcal{G}) = \mathcal{G} \cup [B\colon \exists x \in B \text{ and } \mathcal{U} \subset \psi[x] \cap \mathcal{G}]$. If π is a proximity we require that $h(\pi, \mathcal{G})$ be a grill.

π is called an h-proximity or an RH-proximity iff $h(\pi, \pi(A)) = \pi(A)$ for all $A \subset X$ while $\mathcal{G}$ is called an h-grill with respect to π iff $h(\pi, \mathcal{G}) = \mathcal{G}$. Finally $\mathfrak{M}_h(X, c)$ represents the family of h-proximities which induce the closure operator c on $\mathfrak{X}$.

In view of the fact that all n.p. grills $\mathcal{G}$ are h-grills with respect to $\Delta = \mathcal{G} \times \mathcal{G}$, Theorem 4.3 of [1] with $f = h$ now reads

THEOREM 1. *If $\mathfrak{M}_h(X, c)$ contains a largest element π^* then there exists a largest n.p. h-grill with respect to π^*, say $\mathfrak{D}$, such that*

$$\pi^* = \pi_c^1 \cup (\mathfrak{D} \times \mathfrak{D}) \quad \text{where } \pi_c^1 = \left[(A, B)\colon c(A) \cap c(B) \neq \varnothing\right].$$

Theorems 5.3 and 5.5 of [1] are as follows:

THEOREM 2. *Let X be a regular space and let $\pi \in \mathfrak{M}_h(X, c)$. Then $\mathfrak{D}^B = \mathcal{P}(X) - \mathfrak{B}^B$ is an h-grill with respect to π iff (X, c) is locally bicompact.* ($\mathfrak{B}^B = [C\colon c(C)$ *is bicompact*].)

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THEOREM 3. *Let π be an RH-proximity and let $\mathcal{K}$ be an n.p. h-grill with respect to π; then $\mathcal{K} \subset \mathcal{D}^B$.*

THEOREM 4. *Let $\pi \in \mathfrak{M}_h(X, c)$ and assume that there exists a largest n.p. h-grill $\mathcal{D}$ with respect to π. If $\mathcal{U}$ contains no sets whose closures are bicompact then $\mathcal{U} \subset \mathcal{D}$.*

PROOF. Let $U \in \mathcal{U}$ then $c(U)$ is not bicompact. Hence there exists an ultrafilter $\mathcal{V}$ on $c(U)$ such that $\bigcap [c^*(V): V \in \mathcal{V}] = \varnothing$ where $c^*(V) = c(V) \cap c(U) = c(V)$. Let $\mathcal{W}$ be the ultrafilter on X formed by supersets of $\mathcal{V}$. It follows from Theorem 4.1 of [2] that $\pi(\mathcal{W})$ is an n.p. h-grill with respect to π and hence $\pi(\mathcal{W}) \subset \mathcal{D}$. Therefore $c(U) \in \mathcal{W} \subset \mathcal{D}$.

We claim that the set $\mathcal{S} = [A: \exists a \in A \ni \mathfrak{N}(\pi, [a]) \subset b(\pi, \mathcal{W})]$ is empty (where $\mathfrak{N}(\pi, [a]) = \{N(\pi, [a]): a \in \mathfrak{N}(\pi, [a])$ and $[a] \notin \pi(X - N(\pi, [a]))\}$ and $b(\pi, \mathcal{W}) = [D: c_\pi(D) \in \mathcal{W}]$.

If $\mathcal{S}$ is not empty then there exists $A \in \mathcal{S}$ such that $N(\pi, [a]) \in b(\pi, \mathcal{W})$ for open neighborhoods $N(\pi, [a])$, i.e., $c(N(\pi, [a])) \in \mathcal{W}$. In fact $N(\pi, [a]) \in \mathcal{W}$. Since (X, c) is regular there exists an open neighborhood $N'(\pi, [a])$ such that $N'(\pi, [a]) \subset c(N'(\pi, [a])) \subset N(\pi, [a])$ and since $c(N'(\pi, [a])) \in \mathcal{W}$ it follows that $a \in \bigcap [c(W): W \in \mathcal{W}] \neq \varnothing$. This is false and thus $\mathcal{S} = \varnothing$. Using the equivalent definition for h, viz., $h(\pi, \mathcal{G}) = \mathcal{G} \cup [A: \exists a \in A, \mathfrak{N}(\pi, [a]) \subset \mathcal{G}]$, we therefore have that $h(\pi, b(\pi, \mathcal{W})) = b(\pi, \mathcal{W}) = [B: c(B) \in \mathcal{W}]$. Thus $b(\pi, \mathcal{W}) \subset \mathcal{D}$, i.e., $U \in \mathcal{D}$ and therefore $\mathcal{U} \subset \mathcal{D}$.

THEOREM 5. *If $\mathfrak{M}_h(X, c)$ has a largest element π^* then (X, c) is regular and locally bicompact. π^* is given by $\pi_c^1 \cup (\mathcal{D}^B \times \mathcal{D}^B)$.*

PROOF. Regularity of (X, c) follows from the fact that $\mathfrak{M}_h(X, c) \neq \varnothing$ iff (X, c) is regular. Theorems 1, 3 and 4 show that $\mathcal{D}^B$ is an n.p. h-grill with respect to π^*. That (X, c) is locally bicompact follows from Theorem 2. From Theorem 1 we have that $\pi^* = \pi_c^1 \cup (\mathcal{D}^B \times \mathcal{D}^B)$.

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