CHARACTERIZATION OF \((r, s)\)-ADJACENCY GRAPHS OF COMPLEXES

MARIANNE GARDNER AND FRANK HARARY

Abstract. The \((r, s)\)-adjacency graph of a simplicial complex \(K\) has been defined as the graph whose nodes are the \(r\)-cells of \(K\) with adjacency whenever there is incidence with a common \(s\)-cell. The \((r, s)\)-adjacency graphs for \(r > s\) have been characterized by graph coverings by Dewdney and Harary generalizing the result of Krausz for line-graphs \((r = 1, s = 0)\). We now complete the characterization by handling the case \(r < s\).

1. Introduction. Let \(S\) be a collection of distinct subsets called simplexes of a nonempty finite set \(V\) whose elements are called nodes. Then \(K = (V, S)\) is a (simplicial) complex if it satisfies the condition that every nonempty subset of a simplex \(x \in S\) is also a simplex.

The dimension of a simplex \(x\) in \(K\) is \(r = |x| - 1\) and \(x\) is called an \(r\)-simplex or sometimes an \(r\)-cell. The dimension of a complex \(K\) is the maximum dimension of a simplex in it. A complex of dimension \(r\) is called an \(r\)-complex. Thus a \(1\)-complex is a graph which has at least one line. If the graph is totally disconnected, it is, of course, a \(0\)-complex. A pure \(r\)-complex is one in which every maximal simplex has dimension \(r\).

Every complex \(K\) has an associated hypergraph whose edges are its maximal simplexes. Conversely given any hypergraph, we can construct its complex by including every nonempty subset of an edge as a simplex. Thus an \(r\)-complex is known as an hereditary rank-\(r\) hypergraph.

The \((r, s)\)-adjacency graph, \(r \neq s\), of a complex \(K\) denoted by \(L_{rs}(K)\), in analogy with the standard notation for the line-graph \(L(G)\), is the graph whose nodes are the \(r\)-simplexes of \(K\), with two of these nodes adjacent whenever their \(r\)-simplexes are incident with a common \(s\)-simplex. Thus if \(K\) is a \(1\)-complex, then \(L_{10}(K)\) is its line-graph.

This concept was first suggested by Grünbaum [6] for \(s = r - 1\) and has been investigated for \(r > s\) by Dewdney and Harary [3], by Bermond, Sotteau, Heydemann, Germa in a series [1], [2], [8], and for \(s = 0\) and \(s = r - 1\) by Gardner [4], [5] and others.

Let \(x_1, \ldots, x_t\) be simplexes of \(K\), with no \(x_i\) contained in \(x_j\) for any \(i \neq j\). Their induced complex is the subcomplex \(K'\) whose maximal simplexes are the \(x_i\).

In this statement, condition (i) is redundant because of the partition requirement, but it is useful for later generalization.

**Theorem A (Krausz).** The graph $G$ is a $(1, 0)$-adjacency graph if and only if the edges can be partitioned into a family of complete subgraphs $G_i$ satisfying:

(i) Each edge is in exactly one $G_i$.

(ii) Each vertex is in no more than two $G_i$.

Grünbaum [6] conjectured that this characterization could be extended to graphs which are $(r, r - 1)$-adjacency graphs. Necessary conditions which are not sufficient are given in [2]. When $r > s$, necessary and sufficient conditions for $(r, s)$-adjacency graphs were developed in [3].

**Theorem B (Dewdney and Harary).** The graph $G$ is an $(r, s)$-adjacency graph with $r > s$ if and only if there is a family $G_i$ of subgraphs of $G$ satisfying the following three conditions.

(i) Each edge lies in at most $r$ and at least $s + 1$ of the graphs $G_i$.

(ii) Each vertex lies in at most $r + 1$ of the graphs $G_i$.

(iii) The intersection of any $s + 1$ of the graphs $G_i$ is either empty or a complete graph.

The proof, in analogy with that of Krausz for line-graphs, constructs a complex whose nodes (0-simplexes) are the $G_i$ and which has one $r$-simplex for each node $v$ of $G$. The $r$-simplex corresponding to $v$ contains the 0-simplex $G_i$ whenever node $v$ of $G$ is in subgraph $G_i$.

A few comments are in order. The complete bipartite graph $G = K(2, (2r - 1))$ is the $(2r - 1, r - 1)$-adjacency graph of some complex $K$. Such a complex $K$ can be constructed by taking $4r$ nodes, partitioning them into two sets $x$ and $y$ of size $2r$ each and defining the $(2r - 1)$-simplexes to be $x, y$ and all sets consisting of $r$ nodes from each of $x$ and $y$. It is not possible however to cover the edges of $G$ with a set of complete graphs which satisfies condition (i) of Theorem B. Therefore we cannot expect a characterization of the Krausz type which contains among the $G_i$ a subset consisting of complete graphs which covers the edges of $G$.

Bermond, Heydemann and Sotteau [1] defined the $s$-line-graph of a hypergraph $H$, denoted $L_s(H)$, as the graph whose nodes are the edges of $H$ with two edges adjacent if their intersection contains at least $s$ nodes. Theorem B characterizes the $(s + 1)$-line-graphs of rank-$(r + 1)$ hypergraphs.

In an exact $(r, s)$-adjacency graph, two adjacent simplexes have a common $s$-simplex but not a common $(s + 1)$-simplex. If (i) is replaced by

(i') Each edge lies in exactly $s + 1$ of the graphs $G_i$, then Theorem B characterizes the exact $(r, s)$-adjacency graphs.

We now finish the characterization of all $(r, s)$-adjacency graphs by deriving conditions for $r < s$.

**Theorem 1.** A graph $G$ is the $(r, s)$-adjacency graph of a simplicial complex with $r < s$ if and only if the edges of $G$ are covered by a set of complete subgraphs $G_i$ of...
order \( \binom{n+1}{r} \) which are grouped into subsets \( S_1, \ldots, S_p \) such that the \( G_i \) and \( S_j \) satisfy the following conditions.

(i) The intersection of every subset of the \( G_i \)'s has the form \( K_b \) where \( b \) is a binomial coefficient of the form \( \binom{n+1}{r} \) for some \( n \geq r \). In this case there are exactly \( n + 1 \) sets \( S_j \) containing this subset of \( G_i \)'s.

(ii) Each \( G_i \) is in at most \( s + 1 \) sets \( S_j \).

Proof. Let \( G \) be any graph with isolated nodes \( u_1, \ldots, u_n \). Represent the \( u_k \) by node-disjoint \( r \)-simplexes, one for each \( u_k \). Then \( G \) is an \((r, s)\)-adjacency graph if and only if \( G - \{u_1, \ldots, u_n\} \) is. Assume without loss of generality that \( G \) is connected.

We will first show that the conditions of the theorem are sufficient. We do this by constructing a complex \( K \) with a maximal simplex \( x_i \) for each graph \( G_i \). For each \( G_i \) define disjoint sets \( T_i \) of cardinality \( s + 1 - |\{ S_j : G_i \in S_j \}| \). Set \( x_i = \{ S_j : G_i \in S_j \} \cup T_i \). Let \( K \) be the complex induced by the maximal simplexes \( x_i \).

We prove that \( L_{rs}(K) = G \) by induction on \( t - m \), where \( t \) is the number of \( G_i \)'s. For the induction hypothesis, suppose that there is a one-to-one map from the nodes that lie in at least \( m + 1 \) \( G_i \)'s onto the \( r \)-simplexes of \( K \) incident with at least \( m + 1 \) maximal simplexes, that is, \( s \)-simplexes. In addition suppose that the map has the property that a node \( u \) is mapped to an \( r \)-simplex \( x \), with \( x = \{ S_1, \ldots, S_{r+1} \} \) or \( x = \{ S_1, \ldots, S_r, t_{j+1}, \ldots, t_{r+1} \} \) with \( t_{j+1}, \ldots, t_r \in T_i \) for some \( i \), only if \( S_i \cap \cdots \cap S_{r+1} = \{ G_i : u \in G_i \} \).

Let \( m = 0 \). The intersection of all the \( G_i \)'s is \( K_b \) with \( b = \binom{n+1}{r} \). By condition (i), there are exactly \( n + 1 \) \( S_j \)'s containing all the graphs \( G_i \) and there are therefore exactly \( \binom{n+1}{r} \) \( r \)-simplexes lying in the intersection of all the \( x_i \)'s. Fix any one-to-one map between the nodes of the \( K_b \) and these \( r \)-simplexes.

Now let \( m > 0 \) be given. Fix an \( m \)-set \( \{ G_{i_1}, \ldots, G_{i_m} \} \) with \( H = G_{i_1} \cap \cdots \cap G_{i_m} = K_b, b = \binom{n+1}{r} \). Using condition (i) again, there are exactly \( n + 1 \) \( S_j \)'s containing this subset of \( G_i \)'s and therefore exactly \( \binom{n+1}{r} \) \( r \)-simplexes in \( x_{i_1} \cap \cdots \cap x_{i_m} = \{ S_j, \ldots, S_{s+1} \} \). Some of these \( r \)-simplexes are already the image of a node of \( G \) under the map defined by the induction hypothesis. The number of such \( r \)-simplexes is equal to the number of nodes in the union of the graphs \( H \cap G_{i_{m+1}} \), taken over all \( i_{m+1} \neq i_1, \ldots, i_m \). As this is just the number of \( r \)-simplexes in the corresponding union of the \( x_{i_1} \cap \cdots \cap x_{i_m} \cap x_{i_{m+1}} \), the map can be extended in a one-to-one fashion to map the nodes of \( H = G_{i_1} \cap \cdots \cap G_{i_m} \) to the \( r \)-simplexes of \( x_{i_1} \cap \cdots \cap x_{i_m} \).

Observe that no unassigned node is in two distinct intersections of \( m \) graphs \( G_i \). Thus the map will be well-defined if it is extended in this way and will cover all nodes in any intersection of \( m \) graphs \( G_i \). The map is one-to-one by definition and onto by construction.

Finally we verify that two \( r \)-simplexes are the image of adjacent nodes of \( G \) if and only if the simplexes are incident with a common \( s \)-simplex. Let \( x \) and \( y \) be two \( r \)-simplexes whose preimages in \( G \) are \( u \) and \( v \). If \( u \) and \( v \) are adjacent, then the edge \( uv \) is in some \( G_i \) and \( x \) and \( y \) are both incident with the \( s \)-simplex \( x_i \).
Conversely, if \( x \) and \( y \) are incident with an \( s \)-simplex \( x_i \) then their preimages must be contained in \( G_i \).

To prove the necessity of the conditions, suppose that \( G \) is the \((r, s)\)-adjacency graph of a complex \( K \). Let \( x_i \) be an \( s \)-simplex of \( K \). Then there are \( \binom{r+1}{s+1} \) \( r \)-simplexes in \( x_i \) and the image of \( x_i \) in \( G \) is

\[
K\left( \binom{s+1}{v+1} \right) = G_i.
\]

If \( m \) distinct \( r \)-simplexes overlap on \( n+1 \) points, their images in \( G \) will be a complete graph on \( \binom{n+1}{r+1} \) nodes.

For each \( 0 \)-simplex \( j \) of \( K \), define the set \( S_j = \{ G_i \mid x_i \text{ is an } s \text{-simplex with } j \in x_i \} \). Clearly the \( S_j \) satisfy conditions (i) and (ii). □

REFERENCES

5. ______, Forbidden configurations of large girth for intersection graphs of hypergraphs, Ars Combinatoria (submitted).