**X-INNER AUTOMORPHISMS OF FILTERED ALGEBRAS**

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**Abstract.** For the enveloping algebra of a finite-dimensional Lie algebra, and for the ring of differential polynomials over a commutative domain, we compute the group of those automorphisms which become inner when extended to the quotient division rings. Both of these results depend on a more general result about the automorphisms of a filtered algebra.

Let $A$ be a prime ring. An $X$-inner automorphism $\sigma$ of $A$ is one which becomes inner when extended to the Martindale quotient ring $A_\mathfrak{m}$ of $A$; when $A$ is a prime Goldie ring, this condition is equivalent to $\sigma$ becoming inner on the classical quotient ring of $A$ [5]. Thus the set of all $X$-inner automorphisms is a normal subgroup of Aut($A$) which contains all the inner automorphisms; this set has proved useful in studying group actions on rings and crossed products. Recently the $X$-inner automorphisms have been computed for certain group rings [7] and for coproducts of domains [4].

In this note we consider a filtered algebra $A$ such that the associated graded ring $\tilde{A}$ is a commutative domain. We show that any $X$-inner automorphism of $A$ preserves the filtration of $A$ and induces the trivial automorphism on $\tilde{A}$. We then give two applications of this result: we determine the $X$-inner automorphisms of $U(g)$, the enveloping algebra of a finite-dimensional Lie algebra $g$, and also of the differential polynomial ring $A = R[x; d]$, where $R$ is a commutative domain with nontrivial derivation $d$.

We require some definitions. For a prime ring $A$, the Martindale quotient ring of $A$ is the left quotient ring of $A$ with respect to the filter $\mathfrak{f}$ of all nonzero two-sided ideals; that is $A_\mathfrak{m} = \lim_{\longrightarrow} \text{Hom}_A(A I, A)$. $A_\mathfrak{m}$ is also a prime ring, containing $A$, and for any $0 \neq x \in A$, there exists $I \in \mathfrak{f}$ so that $0 \neq Ix \subseteq A$ (for details see [3]). Moreover, for any $\sigma \in \text{Aut}(A)$, $\sigma$ has a unique extension to $A_\mathfrak{m}$.

Now assume that $\sigma$ is $X$-inner. That is, for some $\sigma \in \text{Aut}(A)$, there exist $0 \neq b, c \in A$ such that $brc = cbr\sigma$, all $r \in A$. Let $I \in \mathfrak{f}$ be such that $0 \neq Ia \subseteq A$, and choose $b, c \in A$ with $0 \neq ba = c$. Then for any $r \in R$, $c(rb)^\sigma = ba(rb)^\sigma = b(rb)a = brc$. That is,

\[(*) \quad brc = crb^\sigma, \quad \forall r \in A.\]

Conversely, it is not difficult to show that if for some $\sigma \in \text{Aut}(A)$, there exist $0 \neq b, c \in A$ such that $brc = crb^\sigma$, all $r \in A$, then $\sigma$ is $X$-inner [6].

Now let $A$ be a filtered ring, say $A = \bigcup_{n \geq 0} A_n$, where $1 \in A_0$. For any $a \in A$, let $f(a)$ denote the filtration of $a$; that is, $f(a) = n$ if $a \in A_n$ but $a \notin A_{n-1}$. The
associated graded ring is \( \widetilde{A} = \sum_{n \geq 0} A_n / A_{n-1} \). We note that \( \widetilde{A} \) is a domain if and only if \( f(ab) = f(a) + f(b) \), all \( a, b \in A \). When \( \widetilde{A} \) is a domain, so is \( A \), and thus \( A_\mathfrak{g} \) is a prime ring.

**Proposition 1.** Let \( A \) be a filtered algebra, as above, such that \( \widetilde{A} \) is a commutative domain. Let \( \sigma \) be an \( A \)-inner automorphism of \( A \). Then:

1. \( \sigma \) preserves the filtration of \( A \) (that is, \( A_n^\sigma = A_n \), all \( n \)),
2. the induced action \( \tilde{\sigma} \) of \( \sigma \) on \( \widetilde{A} \) is trivial.

**Proof.** Since \( \sigma \) is \( A \)-inner, we may choose \( b, c \in A \) such that

\[
brc = cr^b, \quad \text{for all } r \in A. \tag{*}
\]

Since \( \widetilde{A} \) is a domain, \( f(b) + f(r) + f(c) = f(c) + f(r^o) + f(b^o) \) for all \( r \in A \); thus \( f(b) + f(r) = f(r^o) + f(b^o) \). Letting \( r = 1 = r^1 \), \( f(r) = f(r^1) = 0 \), and so \( f(b) = f(b^o) \). It follows that \( f(r) = f(r^o) \), all \( r \in A \). That is, \( \sigma \) preserves the filtration.

Now let \( k = f(b) \), \( l = f(c) \), and let \( \tilde{b} = b + A_{k-1} \), \( \tilde{b}^o = b^o + A_{k-1} \) be the “leading terms” of \( b \) and \( b^o \), respectively. We claim that \( \tilde{b} = \tilde{b}^o \). For, using \( r = 1 \) in (\( * \)), \( bc = cb^o \). Since \( \widetilde{A} \) is commutative, \( cb^o - b^c \in A_{k+l-1} \), and thus \( (b - b^o)c = cb^o - b^o c \in A_{k+l-1} \). Since \( f(c) = l \), it follows that \( f(b - b^o) < k - 1 \). Thus \( b - b^o \in A_{k-1} \), so \( \tilde{b} = \tilde{b}^o \).

Finally, consider any \( x \in A_n \); \( bxc = cx^ob^o \) from (\( * \)), and \( cx^ob^o - b^ox^c \in A_{k+l+n-1} \) since \( \widetilde{A} \) is commutative. Thus \( (bx - b^ox^c) \in A_{k+l+n-1} \); since \( f(c) = l \), we obtain \( f(bx - b^ox^c) < k + n - 1 \), and so \( bx - b^ox^c \in A_{k+n-1} \). Using \( b - b^o \in A_{k-1} \) from above, \( (b - b^o)x^c \in A_{k+n-1} \), and so \( b(x - x^c) = bx - b^ox^c + b^ox^c - bx^c \in A_{k+n-1} \). As before, as \( f(b) = k \); this forces \( x - x^c \in A_{n-1} \). Thus \( \tilde{x} = x^o \), for any \( x \in A_n \). We have proved that the induced automorphism \( \tilde{\sigma} \) (given by \( \tilde{x}^o = x^o \)) is trivial on \( \widetilde{A} \). \( \square \)

Our first application of Proposition 1 is to enveloping algebras. Thus, following [1, §4.3], let \( g \) denote a finite-dimensional Lie algebra over a field \( k \), with universal enveloping algebra \( U = U(g) \). Let \( K = K(g) \) denote the division ring of quotients of \( U \). For \( P \) a completely prime ideal of \( U \), \( A = U/P \) also has a division ring of quotients, which we will denote by \( Q(A) \).

Let \( \epsilon \) be the adjoint representation of \( g \) in \( A \); that is if \( a \in A \) and \( i(x) \) is the image of \( x \in g \) in \( A \), then \( \epsilon(x)a = [i(x), a] \). For \( \lambda \in g^* \), let \( A_\lambda = \{ a \in A | \epsilon(x)a = \lambda(x)a, \text{ all } x \in g \} \). We have \( A_\lambda A_\mu \subset A_{\lambda + \mu} \), and the sum of the \( A_\lambda \) is direct. The subalgebra \( S(A) = \sum_{\lambda \in g^*} A_\lambda \) is called the **semicentre** of \( A \). More generally, as in 4.9.7 of [1], we may also define \( Q(A)_\lambda \).

Give \( A \) the induced filtration from \( U \), and thus \( \epsilon(g) \) generates \( A \). When the associated graded ring \( \widetilde{A} \) of \( A = U/P \) is a domain, we are now able to completely determine those automorphisms of \( A \) which become inner on \( Q(A) \); they are precisely those automorphisms which are given by conjugation by an element of some \( Q(A)_\lambda \).

**Theorem 1.** Let \( A = U(g)/P \), for \( g \) a finite-dimensional Lie algebra, and assume that \( \widetilde{A} \) is a domain. Let \( 0 \neq \alpha \in Q(A) \). Then \( \alpha^{-1}A\alpha = A \leftrightarrow \text{there exists } \lambda \in g^* \text{ such that } a \in Q(A)_\lambda \).
PROOF. First assume that \( a \in Q(A)_\lambda \). Then \( [i(x), a] = \lambda(x)a \), all \( x \in g \), and so \( i(x)a = ai(x) + \lambda(x)a = a(i(x) + \lambda(x)) \). Thus \( i(g)a \subseteq aA \), and it follows that \( Aa \subseteq aA \). Similarly \( aA \subseteq aA \), and so \( aA = Aa \). It follows that \( a^{-1}Aa = A \).

Conversely, if \( a^{-1}Aa = A \), then \( \sigma \in \text{Aut}(A) \) given by \( r^\sigma = a^{-1}ra \) is an \( X \)-inner automorphism of \( A \). By Proposition 1, \( \sigma \) is trivial on the associated graded ring \( A \). In particular, for any \( i(x) \in i(g) \subseteq A_1 \), \( i(x)^\sigma - i(x) \in A_0 = k \cdot 1 \). That is, \( i(x)^\sigma = i(x) + \lambda(x) \), some \( \lambda = \lambda(x) \in k \), for each \( x \in g \). Clearly \( \lambda \in g^* \), and since \( i(x)^\sigma = a^{-1}i(x)a = i(x) + \lambda(x) \), \( [i(x), a] = \lambda(x)a \), all \( x \in g \). Thus \( a \in Q(A)_\lambda \). □

**Corollary 1.** The subgroup of all \( X \)-inner automorphisms of \( A = U(g)/P \), for \( A \) a domain, is isomorphic to the additive subgroup of \( g^* \) consisting of those \( \lambda \) with \( Q(A)_\lambda \neq 0 \).

**Corollary 2.** Consider \( A = U(g)/P \) as above, with \( \overline{A} \) a domain, and let \( \sigma \) be any automorphism of \( A \). Then

1. \( (Q(A)_\lambda)^\sigma = Q(A)_{\mu} \), for some \( \mu \in g^* \).
2. \( \sigma \) stabilizes \( S(A) \), the semicentre of \( A \).

**Proof.** Choose any \( 0 \neq a \in Q(A)_\lambda \). By Theorem 1, \( a^{-1}Aa = A \) and thus \( (a^\sigma)^{-1}A^\sigma a^\sigma = A^\sigma \), or \( (a^\sigma)^{-1}Aa^\sigma = A \). Thus, again by Theorem 1, there exists \( \mu \in g^* \) such that \( a^\sigma \in Q(A)_{\mu} \). It follows that \( Q(A)^{\sigma\mu} = Q(A)_{\mu} \), since \( \sigma \) preserves the center \( C \) of \( Q(A)_\lambda \) and \( Q(A)_{\lambda} = Ca \).

Clearly \( \sigma \) preserves \( S(A) \), since \( A_\lambda = Q(A)_\lambda \cap A \). □

We now turn to derivations of \( U(g) \). The next corollary was pointed out to us by Martha Smith, and we wish to thank her for allowing us to include it.

**Corollary 3 (M. Smith).** Let \( d \) be a derivation of \( g \), a finite-dimensional Lie algebra over a field \( k \) of characteristic 0, and extend \( d \) to \( U(g) \). Then for any \( \lambda \in g^* \) such that \( U_\lambda \neq 0 \),

1. \( d(U_\lambda) \subseteq U_\lambda \),
2. \( \lambda(d(g)) = 0 \).

**Proof.** We first note that by passing to the algebraic closure of \( k \), we may assume that \( k \) is algebraically closed. Since \( d \) is a locally finite derivation on \( U \), and the semicentre \( S = S(U) \) is stable under \( \text{Aut}(U) \) by Corollary 2, we may apply a result of J. Krempa [2] to conclude that \( S \) is also \( d \)-stable.

Let \( 0 \neq a \in U_\lambda \), some \( \lambda \in g^* \). Then \( d(a) \in S \), and so \( d(a) = \sum_{\mu} b_\mu \), where \( b_\mu \in U_\mu \). Applying \( d \) to the equation \( [x, a] = \lambda(x)a \), any \( x \in g \), we see that \( [x, d(a)] = -\lambda(d(x))a + \lambda(x)d(a) \). Now \( \sum \mu(x) b_\mu = \sum \mu(x) b_\mu = [x, b_\mu] = [x, d(a)] = -\lambda(d(x))a + \lambda(x) \sum b_\mu \),

and so

\[ \sum \mu(x) - \lambda(x)) b_\mu = -\lambda(d(x))a. \tag{**} \]

For each \( u \neq \lambda \), this gives \( (\mu(x) - \lambda(x)) b_\mu = 0 \), all \( x \in g \). Since \( \lambda(x_0) \neq \mu(x_0) \) for some \( x_0 \in g \), it follows that \( b_\mu = 0 \), all \( \mu \neq \lambda \). That is, \( d(a) = b_\lambda \in U_\lambda \), proving...
Again using (**) it follows that $0 = -\lambda(d(x))a$. Thus $\lambda(d(x)) = 0$, all $x \in g$, proving (2).

We also give an application to crossed products. For any $k$-algebra $A$, any subgroup $G \subseteq \text{Aut}_k(A)$, and any factor set $t: G \times G \to k$, we may form the crossed product algebra $A \ast_G G$.

**COROLLARY 4.** Let $g$ be a finite-dimensional Lie algebra over a field $k$ of characteristic 0, and let $A = U(g)/p$ as above, such that $A$ is a domain. Let $G$ be any subgroup of $\text{Aut}_k(A)$. Then any crossed product $A \ast_G G$ is a prime ring.

**Proof.** Let $G_{\text{inn}} = \{g \in G | g \text{ is } X\text{-inner}\}$. By Corollary 1, $G_{\text{inn}}$ is isomorphic to a subgroup of the dual space $g^*$, which is torsion-free when $k$ has characteristic 0. Thus the crossed product is prime by [6, Theorem 2.8].

Our second application of Proposition 1 is to certain differential polynomial rings. Let $R$ be a commutative domain with 1 with a nontrivial derivation $d$, and let $A = R[x; d]$, the differential polynomial ring, in which $xr = rx + d(r)$, for all $r \in R$. $A$ is a filtered algebra, using $A_n = \{\text{all polynomials of degree } < n\}$, and the associated graded ring $\overline{A} = R[x]$, the ordinary polynomial ring over $R$. We let $F = \mathbb{Q}(R)$, the quotient field of $R$, and let $D = \mathbb{Q}(A)$, the division ring of quotients of $A$. We are now able to determine all $X$-inner automorphisms of $A$.

**THEOREM 2.** Let $A = R[x; d]$, where $d$ is a nontrivial derivation of the commutative domain $R$, and let $\sigma \in \text{Aut}(A)$. Then $\sigma$ becomes inner on $D = \mathbb{Q}(A) \iff \sigma$ is conjugation by some $q \in F$ such that $q^{-1}d(q) \in R$.

**Proof.** First assume that $\sigma$ is conjugation by some such $q$. Clearly $R^\sigma = R$, and $x^\sigma = q^{-1}xq = q^{-1}(qx + d(q)) = x + q^{-1}d(q) \in A$. Thus $\sigma$ is an $X$-inner automorphism of $A$.

Conversely, assume that $\sigma$ becomes inner on $D$; say that $\sigma$ is induced by $b(x)a(x)^{-1}$, where $a(x) = a_nx^n + \cdots + a_1x + a_0$ and $b(x) = b_nx^n + \cdots + b_0$, $a_i, b_i \in R$. We will show that $\sigma$ is also induced by $q = b_n\sigma^{-1} \in F$ (the fact that $q^{-1}d(q) \in R$ follows from the fact that $\sigma \in \text{Aut}(A)$).

Now by Proposition 1, we know that $\sigma$ preserves the filtration on $A$ and that $\overline{\sigma}$ is trivial on $\overline{A}$. In particular, $r^\sigma = r$, all $r \in R$, and $x^\sigma = x + \alpha$, for some $\alpha \in R$.

To simplify the argument, note that $b(x) = b_1(x)b_n$, so $b(x)a(x)^{-1} = b_1(x)(b_n^{-1}a(x))^{-1}$; that is, we may assume $b(x)$ is monic. Moreover, since for any $f(x) \in A$, $f(x)^\sigma = a(x)b(x)^{-1}f(x)b(x)a(x)^{-1}$, replacing $f$ by $b(x)f$, it follows that $a(x)f(x)b(x) = b(x)^\sigma f(x)^\sigma a(x)$, all $f(x) \in A$. Again since $\overline{\sigma}$ is trivial on $\overline{A}$, $b(x)^\sigma - b(x)$ has degree $< n$; that is, $b(x)^\sigma$ is also monic of degree $n$, say $b(x)^\sigma = c(x) = x^n + \cdots + c_0$. Thus

\[
a(x)f(x)b(x) = c(x)f(x)^\sigma a(x), \quad \text{all } f \in A. \quad (+)
\]

We first evaluate (+) for $f = r = r^\sigma, r \in R$:

\[
(a_mx^m + a_{m-1}x^{m-1} + \cdots + a_0)r(x^n + b_{n-1}x^{n-1} + \cdots + b_0) \\
= (x^n + \cdots + c_0)r(a_mx^m + \cdots + a_0),
\]
so
\[a_m x^{m+n} + a_m md(r)x^{m+n-1} + a_{m-1}x^{m+n-1} + a_m r b_{n-1}x^{m+n-1} + \cdots = ra_m x^{m+n} + nd(ra_m)x^{n+m-1} + c_{n-1}ra_m x^{n+m-1} + ra_{m-1}x^{n+m-1} + \cdots.\]

Equating coefficients of \(x^{n+m-1}\), we see

\[ma_m d(r) + a_{m-1}r + a_m b_{n-1} = nd(r)a_m + nd(a_m) + c_{n-1}ra_m + ra_{m-1}.\]

Cancelling and letting \(r = 1\),

\[nd(a_m) = a_m(b_{n-1} - c_{n-1}).\]

We also evaluate (+) for \(f(x) = x, x^\sigma = x + \alpha:\)

\[(a_m x^m + \cdots + a_0)x(x^\sigma + \cdots + b_0) = (x^n + \cdots + c_0)(x + \alpha)(a_m x^m + \cdots + a_0),\]

\[a_m x^{m+n+1} + a_m b_{n-1} x^{m+n} + a_{m-1}x^{m+n} + \cdots = a_m x^{n+m+1} + (n+1)d(a_m)x^{n+m}
 + c_{n-1}a_m x^{n+m} + a_{m-1}x^{n+m} + \alpha a_m x^{n+m} + \cdots.\]

Equating coefficients of \(x^{n+m}\), we see

\[a_m b_{n-1} + a_{m-1} = (n+1)d(a_m) + c_{n-1}a_m + a_{m-1} + \alpha a_m\]

and thus

\[a_m(b_{n-1} - c_{n-1}) = (n+1)d(a_m) + \alpha a_m.\]

Substituting (++),

\[nd(a_m) = (n+1)d(a_m) + \alpha a_m.\]

Thus \(\alpha a_m = -d(a_m)\), and so \(\alpha = -a_{m-1} d(a_m) = a_m d(a_{m-1})\). Using \(q = a_m^{-1}\), we have proved that

\[x^\sigma = x + \alpha = x + q^{-1}d(q) = q^{-1} xq.\]

Since \(\sigma\) is determined by its action on \(x\), the theorem is proved. \(\square\)

We illustrate Theorems 1 and 2 with several examples.

**Example 1.** Let \(g\) be the 3-dimensional completely solvable Lie algebra over \(k\) with basis \(\{x, y, z\}\) such that \([x, y] = y, [x, z] = z, and [y, z] = 0\). Since \([g, g] = \langle y, z\rangle\), and any \(\lambda \in g^*\) such that \(K_\lambda \neq 0\) must annihilate \([g, g], \lambda(y) = 0 = \lambda(z)\). Thus \(\lambda\) is determined by \(\lambda(x) = \alpha \in k\). We claim that \(\alpha \in \mathbb{Z}\), an integer. For, if \(0 \neq a \in K_\lambda, [y, a] = 0 = [z, a]\) and \([x, a] = aa\). Using \(U(g) = k[y, z][x; d]\), where \(d(y) = y, d(z) = z\), we may assume \(a = pq^{-1},\) where \(p\) and \(q\) are both polynomials in \(y \) and \(z\). Since for a monomial \(m\) in \(y\) and \(z\) of degree \(l, d(m) = lm, d(a) = \alpha a\) implies \(d(p)q - pd(q) = \alpha pq\); it follows that \(\alpha \in \mathbb{Z}\). When \(a = 1\), so \(\lambda(x) = 1\), both \(y, z \in U_\lambda\), and induce the \(X\)-inner automorphism \(\sigma \in Aut(U)\) given by \(x^\sigma = x + 1, y^\sigma = y, z^\sigma = z\). Thus the group of \(X\)-inners of \(U(g)\) is \(\langle \alpha \rangle\), the infinite cyclic group generated by \(\sigma\).

**Example 2.** Consider the Weyl algebra \(A_1 = k[x, y], xy - yx = 1,\) where \(k\) has characteristic 0. Now \(A_1 = k[y][x; d]\) where \(d(y) = 1,\) the usual derivative. Then \(A_1\) has no nontrivial \(X\)-inner automorphisms. For by Theorem 2, if \(\sigma\) is \(X\)-inner, it
is induced by some \( p(y)/q(y) \in k(y) \) such that \( (q(y)/p(y))d(p(y)/q(y)) \in R = k[y] \). Thus, \( (q'p - qp')/pq \) must be a polynomial; this is impossible, as
\[
\text{degree}(pq) > \text{degree}(q'p - qp').
\]
However, enlarging \( A_1 \) slightly to \( B = k[y]_{y; q} \), the localization of \( A_1 \) at the powers of \( y \), the group of \( A' \)-inners of \( B \) is \( \langle \sigma \rangle \), the infinite cyclic group generated by \( \sigma \) which is given by conjugation by \( y \). For, by Theorem 2, any \( A' \)-inner automorphism \( \tau \) is induced by \( py^k/q \in k(y) \), where \( p, q \in k[y] \) are relatively prime, \( y \nmid p, y \nmid q \), and \( k \in \mathbb{Z} \). Since conjugation by \( y^k \) is easily seen to be \( A' \)-inner (it preserves \( k[y]_{y;p} \)) and the \( A' \)-inner automorphisms form a subgroup, \( \tau_1 = \sigma^{-k} \) is \( A' \)-inner and is induced by \( p/q \). As before, \( (q/p)d(p/q) = (q'p - qp')/pq \in k[y]_{y;p} \), and this is impossible as \( y \nmid pq \), unless \( q' = p' = 0 \), so \( p/q = \alpha \in k \). Thus, \( \tau \) is induced by \( y^k \).

Finally, if we consider \( C = k(y)[x; d] \), then conjugation by any nonzero element of \( F = k(y) \) is an \( A' \)-inner automorphism of \( C \), and \( p, q \in F \) induce the same automorphism \( \iff p = \alpha q, \alpha \in k \). Thus the group of \( A' \)-inners is isomorphic to \( k(y)^o/k^o \), where \( K^o \) denotes the multiplicative group.

**References**


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