**Absolutely Convergent Fourier Series of Distributions**

NICOLAS K. ARTEMIAIDIS

Abstract. Let $S$ be a distribution (in the sense of L. Schwartz) defined on the circle $T$, and suppose that $S$ is equal to a function in $L^\infty$ on an open interval of $T$. A necessary and sufficient condition is given in order that the Fourier series of $S$ converges absolutely.

1. Introduction. The problem of characterizing the class of all Lebesgue integrable complex-valued functions on the circle $T$ (the additive group of the reals modulo $2\pi$) is a very important one in the theory of Fourier series. In [1] (see also [2]), we gave criteria for a function $f \in L^1(T)$ to have an absolutely convergent Fourier series. The criteria given in [1] are to be compared with those given by M. Riesz and S. B. Stečkin (see [2]). It seems that one of the useful aspects of the method used in [1] is that it can be extended to the case where $f$ is a distribution, and this is exactly what we intend to do in this paper. I am indebted to Paul Malliavin for helpful discussion on the subject.

2. Preliminaries and notation. With $C^\infty$ we denote the set of all $2\pi$-periodic infinitely differentiable functions. Let $\varphi$ be a distribution defined on $T$, which is equal to a function of the class $V_0$ on an open interval $I$ of $T$ containing the point $a \in \mathbb{R}$. For each $\varphi \in C^\infty$ set $\langle S, \varphi \rangle = \langle S, \varphi_a \rangle$ where $\varphi_a(t) = \varphi(t - a)$, $a \in \mathbb{R}$. Clearly $S_a$ is also a distribution which is equal to a function of $L^\infty$ in an open interval containing the origin. By $S_0$ we mean the distribution $S$.

The Fourier coefficients of $S_a$ are

$$\hat{S}_a(n) = \langle S_a, e^{-int} \rangle = \langle S, e^{-int(a)} \rangle = e^{ina} \langle S, e^{-int} \rangle = e^{ina} \hat{S}(n).$$

For $b \in \mathbb{R}$ we define, as usual, $b^+ = \max(b, 0)$, $b^- = \max(-b, 0)$. $\Re z$, $\Im z$ mean the real and imaginary parts of $z$ respectively.

3. The main theorem.

Theorem. Let $S$ be a distribution (in the sense of L. Schwartz) defined on $T$. Suppose that on an open interval $I$ of $T$, $S$ is equal to a function of $L^\infty$. Then the Fourier series of $S$ is absolutely convergent if, for some $a \in I$, the sequences

$$(1) \quad <(\Re \hat{S}_a(n))^\infty_{n=1}, \quad (\Im \hat{S}_a(n))^\infty_{n=1},$$

both belong to $l^1$.

The converse of the above statement also holds.
Proof. Suppose the sequences in (1) belong to $l^1$. We first consider the case $\alpha = 0$. Let $\varphi$ be an infinitely differentiable function with support in the interval $(-\pi, \pi)$, and such that $\hat{\varphi}(t) > 0$, $\hat{\varphi}(0) = 1$, where $\hat{\varphi}$ is the Fourier transform of $\varphi$.

Set

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}.$$  

For sufficiently small $\varepsilon > 0$, the function $\varphi_\varepsilon$ is also infinitely differentiable with support in $(-\pi, \pi)$. Then, $\varphi_\varepsilon$ can be extended to a $2\pi$-periodic function $\tilde{\varphi}_\varepsilon \in C^\infty$. Put $\tilde{\varphi}_\varepsilon \ast S = u_\varepsilon$. It is known (see [3, p. 71]) that $u_\varepsilon \in C^\infty$. Hence $u_\varepsilon$ equals the sum of its Fourier series. We have

$$u_\varepsilon(0) = \sum_n \hat{\varphi}(en)\hat{S}(n) = \lim_{N \to \infty} \sum_{n=-N}^N \hat{\varphi}(en)\hat{S}(n).$$

Set

$$\sigma_{N,\varepsilon} = \sum_{n=-N}^N \hat{\varphi}(en)\hat{S}(n) = \sum_{n=-N}^N \hat{\varphi}(en)[\text{Re}(\hat{S}(n)) + i\text{Im}(\hat{S}(n))]$$

$$= \sum_{n=-N}^N \hat{\varphi}(en)[(\text{Re} \, \hat{S}(n))^+ - (\text{Re} \, \hat{S}(n))^- + i(\text{Im} \, \hat{S}(n))^+ - i(\text{Im} \, \hat{S}(n))^+]$$

$$= \sum_{n=-N}^N \hat{\varphi}(en)(\text{Re} \, \hat{S}(n))^+ + i\sum_{n=-N}^N \hat{\varphi}(en)(\text{Im} \, \hat{S}(n))^+$$

$$- \sum_{n=-N}^N \hat{\varphi}(en)(\text{Re} \, \hat{S}(n))^- - i\sum_{n=-N}^N \hat{\varphi}(en)(\text{Im} \, \hat{S}(n))^-.$$

Next observe that, due to the hypothesis that $S$ equals a function of $L^\infty$ in $I$, $u_\varepsilon(0)$ remains uniformly bounded when $\varepsilon \to 0$. Furthermore since, by assumption, the sequences in (1) belong to $l^1$, it follows that the expressions

$$\sum_{n=-N}^N \hat{\varphi}(en)(\text{Re} \, \hat{S}(n))^+, \quad \sum_{n=-N}^N \hat{\varphi}(en)(\text{Im} \, \hat{S}(n))^-$

remain uniformly bounded for all $\varepsilon$ and $N$. Set

$$A_{m,n} = \hat{\varphi}\left(\frac{n}{m}\right)(\text{Re} \, \hat{S}(n))^+,$$

where $m, n$ are natural numbers.

It follows from what we have just proved that the double series $\sum_{m,n} A_{m,n}$ is convergent since $A_{m,n} > 0$. We have

$$\sum_{m,n} A_{m,n} = \lim_{N \to \infty} \left(\lim_{m \to \infty} \sum_{n=-N}^N \hat{\varphi}\left(\frac{n}{m}\right)(\text{Re} \, \hat{S}(n))^+\right)$$

$$= \lim_{N \to \infty} \sum_{n=-N}^N (\text{Re} \, \hat{S}(n))^+ = \sum_{n \in \mathbb{Z}} (\text{Re} \, \hat{S}(n))^+ < +\infty.$$

In a similar way we prove that

$$\sum_{n \in \mathbb{Z}} (\text{Im} \, \hat{S}(n))^+ < +\infty.$$
It follows that
\[ \sum_{n \in \mathbb{Z}} |\hat{S}(n)| < +\infty. \]
This proves the theorem in the case \( \alpha = 0 \).

Next assume \( \alpha \neq 0 \) and consider the distribution \( S_\alpha \). As we have noticed before, \( S_\alpha \) is a distribution which is equal to a function of \( L^\infty \) in an interval containing the origin, so that the above result holds for \( S_\alpha \). We have
\[ \sum_{n \in \mathbb{Z}} |\hat{S}_\alpha(n)| = \sum_{n \in \mathbb{Z}} |e^{i\alpha n}\hat{S}(n)| = \sum_{n \in \mathbb{Z}} |\hat{S}(n)| < +\infty. \]
This proves the theorem.

To prove the converse, let \( S \) be a distribution on \( T \) such that \( \sum_{n \in \mathbb{Z}} |\hat{S}(n)| < +\infty \). The sequence \( \langle \hat{S}(n) \rangle_{n \in \mathbb{Z}} \) being tempered (see [3, p. 65]) the distributions \( S_N = \sum_{|n| < N} \hat{S}(n) e_n \) (where \( e_n \) is the function \( x \rightarrow e^{in\lambda} \) converge in the space of distributions, as \( N \to \infty \), to a distribution \( F \), so that \( \hat{F}(n) = \hat{S}(n) \) \((n \in \mathbb{Z}) \)). Hence \( F = S \). Now the function \( f(x) = \sum_{n \in \mathbb{Z}} \hat{S}(n) e^{in\lambda} \) can be considered as a distribution. It is easily seen that, due to the uniform convergence of the last series, we have, for each \( \varphi \in C^\infty \),
\[ (S, \varphi) = \lim_{N \to \infty} (S_N, \varphi) = (1/2\pi) \int f(x)\varphi(x) \, dx = (f, \varphi), \]
which shows that \( S \) is an \( L^\infty \) function on \( T \).

One might find the above result interesting, also because of the following remark (see [1], [2]).

**Remark.** Call a numerical series \( \sum (a_n + ib_n) \) \((a_n, b_n \in \mathbb{R}) \) "one sidedly absolutely convergent" (O.A.C.), iff:

(at least one of \( \sum a_n^+ \), \( \sum a_n^- \)) and (at least one of \( \sum b_n^+ \), \( \sum b_n^- \)) is finite.

Now it is possible that a series \( \sum (a_n + ib_n) \) is not O.A.C. while the series \( \sum (a_n + ib_n)e^{i\lambda} \) is O.A.C. In other words a non-O.A.C. series can, in some cases, be converted to an O.A.C. series by just multiplying each term by a factor of the form \( e^{i\lambda} \) \((\lambda = \text{some constant}) \) or perhaps in some other way.

**Example.** Let \( c_n = a_n + ib_n \), where \( c_{2n} = 1 + i, c_{2n+1} = 1 - i, n = 0, 1, 2, \ldots \), and \( \lambda = \pi/4 \). Then it is easily seen that \( \sum c_n \) is not O.A.C. while \( \sum c_n e^{i\lambda} \) is.

The theorem we have just proved essentially says that, the Fourier series of a distribution converges absolutely iff \( \sum \hat{S}_\alpha(n) \) is O.A.C. for some \( \alpha \in \mathbb{R} \).

**References**


**Department of Mathematics, University of Patras, Patras, Greece**