

A DIRICHLET NORM INEQUALITY AND SOME INEQUALITIES FOR REPRODUCING KERNEL SPACES

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ABSTRACT. Let f be analytic and of finite Dirichlet norm in the unit disk Δ with $f(0) = 0$. Then, for any $q > 0$,

$$\|\exp f\|_q^2 < \exp \left\{ \frac{1}{\pi q} \int_{\Delta} |f'(z)|^2 d\sigma(z) \right\} \quad (d\sigma(z) \equiv (i/2) dz \wedge d\bar{z}).$$

Equality holds if and only if $f(z) = -q \log(1 - z\bar{\zeta})$ for some $\zeta \in \Delta$. Here, for $g(z) = \sum_{n=0}^{\infty} b_n z^n$, analytic in Δ ,

$$\|g\|_q^2 \equiv \sum_{n=0}^{\infty} \frac{n!}{(q)_n} |b_n|^2,$$

where $(q)_0 = 1$ and $(q)_n = q(q+1) \cdots (q+n-1)$ for $n > 1$. This also extends with a substantially easier proof, a result of Saitoh concerning the case of $q > 1$. In addition, a sharp norm inequality, valid for two functional Hilbert spaces whose reproducing kernels are related via an entire function with positive coefficients, is established.

1. Introduction. Let Δ be the unit disk in the complex plane and let q be a fixed positive real number. We shall write $d\sigma(z) = 2^{-1}i dz \wedge d\bar{z}$ for the area Lebesgue measure. Quite recently, Saitoh [5] has established the following result.

THEOREM A. Assume $q \geq 1$ and let f be an analytic function in Δ with a finite Dirichlet norm and $f(0) = 0$. Then

$$(1.1) \quad \frac{q-1}{\pi} \int_{\Delta} |\exp f(z)|^2 (1 - |z|^2)^{q-2} d\sigma(z) < \exp \left\{ \frac{1}{\pi q} \int_{\Delta} |f'(z)|^2 d\sigma(z) \right\};$$

$q > 1$,

and

$$(1.2) \quad \frac{1}{2\pi} \int_{\partial\Delta} |\exp f(z)|^2 |dz| < \exp \left\{ \frac{1}{\pi} \int_{\Delta} |f'(z)|^2 d\sigma(z) \right\} \quad (q = 1).$$

Equality holds if and only if $f(z)$ is of the form $-q \log(1 - z\bar{\zeta})$ for some $\zeta \in \Delta$.

Saitoh [5] obtained this result by appealing to the general theory of reproducing kernels of Aronszajn [1]. His proof concerning the equality statement, however, is difficult and rather lengthy. In this paper we shall give a shorter proof of Theorem A and in so doing even extend the theorem to the case of $0 < q < 1$. In the extreme case of $0 < q < 1$, which is not covered by Saitoh [5], the integrals on the

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left-hand sides of (1.1) or (1.2) have to be replaced by a suitable Hardy-space norm. We shall also establish a sharp inequality which is valid for general reproducing kernel spaces.

2. A Dirichlet norm inequality. We denote by $H(\Delta)$ the class of all analytic functions in the unit disk Δ . For $q > 0$, we write

$$D_q = \{f \in H(\Delta): f(0) = 0, \|f\|_{D_q} < \infty\}$$

with

$$(2.1) \quad \|f\|_{D_q}^2 = \frac{1}{\pi q} \int_{\Delta} |f'(z)|^2 d\sigma(z) = \frac{1}{q} \sum_{n=1}^{\infty} n |a_n|^2$$

and where

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in \Delta.$$

We shall call D_q the q -Dirichlet-space, and we observe that it is a functional Hilbert space of analytic functions in Δ with the reproducing kernel

$$k_q(z, \bar{\zeta}) = -q \log(1 - z\bar{\zeta}), \quad z, \zeta \in \Delta.$$

Another related functional Hilbert space is the so-called (generalized) q -Hardy-space H_q . This is the space of analytic functions g in Δ whose norm is given by

$$(2.2) \quad \|g\|_q^2 = \sum_{n=0}^{\infty} \frac{n!}{(q)_n} |b_n|^2; \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \Delta.$$

Here $(q)_0 = 1$ and $(q)_n = q(q+1) \cdots (q+n-1)$ for $n > 1$. The reproducing kernel of H_q is $K_q(z, \bar{\zeta}) = (1 - z\bar{\zeta})^{-q}$, and we note the relationship $K_q(z, \bar{\zeta}) = \exp[k_q(z, \bar{\zeta})]$, $z, \zeta \in \Delta$.

The space H_q , $q > 0$, can be also defined for any simply-connected domain ($\neq \mathbb{C}$) via an isometry induced by conformal mappings (see [3] for more details). Furthermore, the norm in (2.2) can be identified as an L_2 -norm of a certain Laplace transform (see [3]). For $q > 1$ the norm can be also realized in a form which appears frequently in function theory. In fact

$$\|g\|_q^2 = \frac{q-1}{\pi} \int_{\Delta} |g(z)|^2 (1 - |z|^2)^{q-2} d\sigma(z), \quad q > 1,$$

$$\|g\|_1^2 = \frac{1}{2\pi} \int_{\partial\Delta} |g(z)|^2 |dz|,$$

where in the last integral, g stands for the nontangential boundary values of the analytic function $g(z)$ in Δ . Thus, H_1 is the usual Hardy-Szegö space, H_2 is the Bergman space and H_q , $q > 1$, is the Bergman-Selberg space [3]. The norm of H_q , $0 < q < 1$, does not admit such a simple realization. However, this fact is not crucial for we shall only use (2.2) as the norm of $g \in H_q$, $q > 0$.

The following, therefore, constitutes an extension of Theorem A of Saitoh [5].

THEOREM 1. *Let $q > 0$ and let $f \in D_q$. Then*

$$\|\exp f\|_q^2 < \exp \|f\|_{D_q}^2$$

with equality holding if and only if f is of the form $f(z) = k(z, \bar{\zeta}) = -q \log(1 - z\bar{\zeta})$ for some $\zeta \in \Delta$.

The main difficulty in this theorem is in the necessity of the equality statement. The proof of Saitoh [5], who confined himself only to the case of $q > 1$, is rather difficult and involved. We shall give another proof of Theorem 1 which is easier than that of Saitoh [5] and also extends it to $0 < q < 1$.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and let

$$(2.3) \quad g(z) = \exp[f(z)] = \sum_{n=0}^{\infty} b_n z^n$$

be its exponential transform. Then

$$(2.4) \quad b_n = \frac{1}{n} \sum_{k=0}^{\infty} k a_k b_{n-k}; \quad n = 1, 2, \dots \quad (b_0 = 1).$$

3. Proof of Theorem 2. In view of (2.1), (2.2), (2.3) and (2.4), Theorem 1 is equivalent to the following lemma.

LEMMA 1. *Let the notation of (2.3) and (2.4) apply and let $q > 0$. Then*

$$(3.1) \quad \sum_{n=0}^{\infty} \frac{n!}{(q)_n} |b_n|^2 \leq \exp \left\{ \frac{1}{q} \sum_{n=1}^{\infty} n |a_n|^2 \right\},$$

if the right-hand side is finite. Equality holds if and only if $a_n = qn^{-1}\bar{\zeta}^n$ ($n > 1$) and $b_n = (n!)^{-1}(q)_n \bar{\zeta}^n$ ($n > 0$) for some $\zeta \in \Delta$.

PROOF. For $r \in [0, 1)$, define

$$A(r) = \frac{1}{q} \sum_{n=1}^{\infty} n |a_n|^2 r^n, \quad B(r) = \sum_{n=0}^{\infty} \frac{n!}{(q)_n} |b_n|^2 r^n; \quad r \in [0, 1).$$

By (2.4) and the Cauchy-Schwarz inequality

$$\begin{aligned} |b_n|^2 &= \frac{1}{n^2} \left| \sum_{k=1}^n k a_k \left[\frac{(n-k)!}{(q)_{n-k}} \right]^{1/2} b_{n-k} \left[\frac{(q)_{n-k}}{(n-k)!} \right]^{1/2} \right|^2 \\ &\leq \frac{1}{n^2} \left(\sum_{k=1}^n k^2 |a_k|^2 \frac{(n-k)!}{(q)_{n-k}} |b_{n-k}|^2 \right) \sum_{k=1}^n \frac{(q)_{n-k}}{(n-k)!} \quad (n > 1). \end{aligned}$$

On the other hand we have the identity

$$(3.2) \quad \sum_{k=1}^n \frac{(q)_{n-k}}{(n-k)!} = \sum_{k=0}^{n-1} \frac{(q)_k}{k!} = \frac{1}{q} \frac{(q)_n}{(n-1)!}$$

and therefore

$$(3.3) \quad n \frac{n!}{(q)_n} |b_n|^2 < \frac{1}{q} \sum_{k=1}^n k^2 |a_k|^2 \frac{(n-k)!}{(q)_{n-k}} |b_{n-k}|^2 \quad (n > 1).$$

This shows that $B'(r) < A'(r)B(r)$ or $[\log B(r)]' < A'(r)$. Therefore, since $A(0) = 0$ and $B(0) = 1$,

$$\log B(r) = \int_0^r [\log B(x)]' dx < \int_0^r A'(x) dx = A(r)$$

and (3.1) follows by letting $r \rightarrow 1 - 0$. In view of $[\log B(r)]' < A'(r)$, $r \in [0, 1)$, equality in (3.1) holds if and only if $B'(r) = A'(r)B(r)$ for each $r \in [0, 1)$ which is equivalent to having equality in (3.3) for every $n > 1$. This is, clearly, equivalent to an existence of $\lambda_n \in \mathbb{C}$ so that

$$(3.4) \quad ka_k b_{n-k} = \lambda_n \frac{(q)_{n-k}}{(n-k)!}; \quad k = 1, 2, \dots, n.$$

Putting $k = n$ in (3.4) results in $\lambda_n = na_n$. Also, using (2.4), (3.4) and the identity (3.2), we have $\lambda_n = n!b_n q / (q)_n$ and therefore

$$(3.5) \quad na_n = q \frac{n!}{(q)_n} b_n; \quad n = 1, 2, \dots$$

The only solution of (3.5) subject to (2.4) is

$$(3.6) \quad a_n = \frac{q}{n} \bar{\zeta}^n, \quad \bar{\zeta} \equiv \frac{a_1}{q}; \quad n = 1, 2, \dots,$$

and, therefore, $b_n = (q)_n \bar{\zeta}^n / n!$ for $n \geq 0$. Indeed, for $n \geq 1$, in view of (2.4) and (3.5),

$$\begin{aligned} na_n &= q \frac{n!}{(q)_n} \frac{1}{n} \sum_{k=0}^{n-1} (n-k) a_{n-k} b_k \\ &= q \frac{n!}{(q)_n} \frac{1}{n} \left[na_n + \sum_{k=1}^{n-1} (n-k) a_{n-k} b_k \right] \\ &= q \frac{n!}{(q)_n} \frac{1}{n} \left[na_n + \frac{1}{q} \sum_{k=1}^{n-1} \frac{k(n-k)}{k!} a_k a_{n-k} (q)_k \right]. \end{aligned}$$

This results in

$$(3.7) \quad a_n \left[n - n! \frac{q}{(q)_n} \right] = \frac{n!}{n} \frac{1}{(q)_n} \sum_{k=1}^{n-1} \frac{k(n-k)}{k!} a_k a_{n-k} (q)_k.$$

To prove (3.6), we use induction on n . For $n = 1$, (3.6) is a triviality while for $n > 1$, by (3.7) and the inductive assumption,

$$a_n \left[n - n! \frac{q}{(q)_n} \right] = \frac{n!}{n} \frac{1}{(q)_n} q^{2\bar{\zeta}n} \sum_{k=1}^{n-1} \frac{(q)_k}{k!}, \quad n > 1.$$

Therefore, by (3.2),

$$a_n \left[n - n! \frac{q}{(q)_n} \right] = \frac{n!}{n} \frac{1}{(q)_n} q^{2\bar{\zeta}n} \left[\frac{1}{q} \frac{(q)_n}{(n-1)!} - 1 \right], \quad n > 1,$$

or

$$a_n \left[n - n! \frac{q}{(q)_n} \right] = \frac{q}{n} \bar{\zeta}^n \left[n - n! \frac{q}{(q)_n} \right], \quad n > 1.$$

However, the common factor in the square bracket is positive for $n > 1$ and $q > 0$, and (3.6) follows at once. Since the right-hand side of (3.1) is finite, ζ of (3.6) must satisfy $|\zeta| < 1$. This concludes the proof of the lemma.

4. Extensions to reproducing kernel spaces. The inequality and the sufficient condition in the equality statement of Theorem 1 exhibit a phenomenon which is valid in any two functional Hilbert spaces whose reproducing kernels are related via an entire function with positive coefficients. We shall briefly discuss this possibility, keeping in mind Theorem 1 as a motivation.

Let $k = k(z, \bar{\zeta})$ be a positive-definite Hermitian kernel where z, ζ vary in a fixed base set D . Then, as is well known [1], the kernel k determines a uniquely defined Hilbert space H_k of functions on D with norm $\|f\|_k$ and for which k is its reproducing kernel. Although not necessary, we shall always assume that k is truly Hermitian in the sense that it is complex-valued. In this case, H_k is a Hilbert space over the complex field \mathbb{C} .

Let F be a nonconstant entire function

$$F(t) = \sum_{n=0}^{\infty} r_n t^n; \quad r_n > 0,$$

and consider the positive-definite Hermitian kernel

$$K(z, \bar{\zeta}) = F[k(z, \bar{\zeta})] = \sum_{n=0}^{\infty} r_n [k(z, \bar{\zeta})]^n; \quad z, \zeta \in D,$$

on D . This kernel determines another Hilbert space H_K of functions g on D with norm $\|g\|_K$ and for which K is its reproducing kernel. The relationship between these two Hilbert spaces is

$$H_K = F(H_k) = \bigoplus_{n=0}^{\infty} r_n [(H_k)_{\otimes}^n]_r,$$

where $[(H_k)_{\otimes}^0]_r = (H_k)_{\otimes}^0 = \mathbb{C}$, $[(H_k)_{\otimes}^1]_r = (H_k)_{\otimes}^1 = H_k$ and $[(H_k)_{\otimes}^n]_r, n \geq 2$, is described below. Let $\{\phi_{\alpha}\}, \alpha \in \Lambda$, be an orthonormal basis for H_k , then the space $(H_k)_{\otimes}^n, n \geq 2$, is the Hilbert space of functions f on D^n

$$f(z_1, z_2, \dots, z_n) = \sum A_{\alpha_1 \alpha_2 \dots \alpha_n} \phi_{\alpha_1}(z_1) \phi_{\alpha_2}(z_2) \cdots \phi_{\alpha_n}(z_n)$$

with the norm

$$\|f\|_{k: \otimes n} = \left\{ \sum |A_{\alpha_1 \alpha_2 \dots \alpha_n}|^2 \right\}^{1/2} < \infty; \quad (\alpha_1, \alpha_2, \dots, \alpha_n) \in \Lambda^n.$$

The space $[(H_k)_{\otimes}^n]_r (n \geq 2)$ is formed by restricting functions in $(H_k)_{\otimes}^n$ to the diagonal set of D^n . Here, the norm of $f \in [(H_k)_{\otimes}^n]_r$ is given by $\min \|h\|_{k: \otimes n}$ for all h , whose restriction to the diagonal of D^n is f . For additional details, see Aronszajn [1] (see also Saitoh [5]).

An element $g \in H_K$ can therefore be written as

$$(4.1) \quad g(z) = \sum_{n=0}^{\infty} r_n f_n(z); \quad z \in D,$$

where $f_0(z)$ is a constant, $f_1 \in H_k$ and $f_n \in [(H_k)_{\otimes}^n]_r$. The norm is given by

$$(4.2) \quad \|g\|_K^2 = \min \left\{ \sum_{n=0}^{\infty} r_n \|f_n\|_{k: \otimes n, r}^2 \right\}$$

where the minimum is taken over all representations (4.1) of $g \in H_K$. Let now $f \in H_k$. Then

$$F[f(z)] = \sum_{n=0}^{\infty} r_n [f(z)]^n, \quad z \in D,$$

and clearly $[f(z)]^n \in [(H_k)_{\otimes}^n]_r, n > 0$. Consequently, using (4.2), we obtain

$$\|F \circ f\|_K^2 < \sum_{n=0}^{\infty} r_n \|f\|_k^{2n} = F(\|f\|_k^2).$$

We therefore have the following theorem.

THEOREM 2. *Let F be a nonconstant entire function with nonnegative coefficients. Let H_k and H_K be Hilbert spaces of functions on the same base set D with reproducing kernels k and K , respectively. Assume further that $K(z, \zeta) = F[k(z, \zeta)]$ for every $z, \zeta \in D$. Then*

$$(4.3) \quad \|F \circ f\|_K^2 < F(\|f\|_k^2); \quad f \in H_k.$$

Equality in (4.3) occurs when f is of the form $f(z) = k(z, \zeta)$ for all $z \in D$ and some $\zeta \in D$.

In order to prove the necessity of the equality statement one must add some additional structure to the Hilbert spaces involved. One such a requirement is, for example, that H_k is a space of analytic functions in a bounded domain D of \mathbb{C}^n . One is then able to choose a particularly convenient orthonormal basis. Even with these assumptions the establishment of the necessity of the equality statement is still technically difficult (see [2], [4] for details). We shall not pursue this point in its full generality. We, however, note that Theorem 1 is specialized here in the assumptions that D is the unit disk Δ and F is the entire function $F(t) = \exp(t)$. The last theorem may also be applied to spaces of square summable series described below.

Let $\Delta_\rho = \{z \in \mathbb{C}: |z| < \rho\}, \rho = 1, \infty$, and assume that $\phi(z) = \sum_{n=0}^{\infty} c_n z^n, c_n > 0$, is analytic in Δ_ρ . Let $H(\Delta_\rho)$ denote the class of analytic functions in Δ_ρ and write

$$H_\phi = \left\{ f \in H(\Delta_\rho): f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} c_n^{-1} |a_n|^2 < \infty \right\}.$$

Clearly, this is a Hilbert space with the norm

$$\|f\|_\phi = \left\{ \sum_{n=0}^{\infty} c_n^{-1} |a_n|^2 \right\}^{1/2}.$$

We have

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n = \sum_{n=0}^{\infty} \frac{a_n c_n \zeta^n}{c_n} = (f(z), \phi(z\bar{\zeta}))_{\phi}$$

which means that $K(z, \bar{\zeta}) = \phi(z\bar{\zeta})$ is the reproducing kernel of H_{ϕ} . The sequence $\{\sqrt{c_n} z^n\}$, $n \geq 0$, is, of course, an orthonormal basis for H_{ϕ} . It is also clear that $\phi_m(z\bar{\zeta}) = \sum_{n=m}^{\infty} c_n (z\bar{\zeta})^n$, $m \geq 1$, is the reproducing kernel of the closed subspace of H_{ϕ}

$$H_{\phi_m} = \{f \in H_{\phi} : f(0) = f'(0) = \dots = f^{(m-1)}(0) = 0\}.$$

Therefore, by making the identification $\phi_0 = \phi$, we may allow some of the c_n in the definition of H_{ϕ} to be zero with the obvious interpretation.

With this notation, Theorem 2 admits the following corollary.

COROLLARY 1. *Let F be a nonconstant entire function with nonnegative coefficients. Let ϕ be as before and write*

$$\psi(z) = F[\phi(z)] = \sum_{n=0}^{\infty} d_n z^n \quad (z \in \Delta_{\rho}, d_n > 0).$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H_{\phi}$ and consider its transform

$$F[f(z)] = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \Delta_{\rho}.$$

Then $F \circ f \in H_{\psi}$ and

$$(4.4) \quad \sum_{n=0}^{\infty} d_n^{-1} |b_n|^2 < F\left(\sum_{n=0}^{\infty} c_n^{-1} |a_n|^2\right).$$

Equality occurs when $f(z) = \phi(z, \bar{\zeta})$ for some $\zeta \in \Delta_{\rho}$ or when $a_n = c_n \bar{\zeta}^n$ ($b_n = d_n \bar{\zeta}^n$).

Many interesting inequalities arise from (4.4) by a suitable choice of the functions ϕ and F . These inequalities are then interpreted as norm inequalities for Hilbert spaces of entire functions when $\rho = \infty$ or of analytic functions in Δ when $\rho = 1$ (see [3] for details). As an example, we take the unit disk Δ and $q > 0$. We choose $\phi(z) = -q \log(1 - z)$ and, therefore, $c_0 = 0$ and $c_n = q^{n-1}$ for $n \geq 1$. The Hilbert space H_{ϕ} induced by ϕ is then the q -Dirichlet-space D_q whose norm is as in (2.1), and, $k_q(z, \bar{\zeta}) = \phi(z\bar{\zeta})$. We also take $F(t) = \exp(t)$ and therefore $\psi(z) = (1 - z)^{-q}$. The Hilbert space H_{ψ} is now the q -Hardy-space H_q whose norm is as in (2.2), and $K_q(z, \bar{\zeta}) = \psi(z\bar{\zeta})$. The assertions of Theorem 1, short of the necessity of the equality statement, are clearly deductible from Corollary 1.

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