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PARTS IN H^∞ WITH HOMEOMORPHIC ANALYTIC MAPS

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ABSTRACT. Hoffman characterized the parts of H^∞ as either singleton points or analytic discs. He showed that a part belongs to the latter category if and only if it is hit by the closure of an interpolating sequence and that there are cases where a corresponding analytic map is a homeomorphism and cases where it is not. We show that there is no class, $\mathcal{C}$, of subsets of the open unit disc such that an analytic map of a part P is a homeomorphism if and only if P is hit by the closure of some set in $\mathcal{C}$.

1. Introduction and preliminaries. We assume the reader is familiar with the theory of H^∞ and especially Hoffman's paper [1]. The setting of this paper will be mainly in H^∞ of the open right half plane, H, but will make use of the obvious correspondence with H^∞ of the open unit disc D.

In [1] Hoffman found the striking characterization of nontrivial parts as those hit by the closure in the maximal ideal space of H^∞ of some interpolating sequence. Given a nontrivial part P and $h_0 \in P$ we let τ_{h_0} denote that analytic map of H onto P such that $\tau_{h_0}(1) = h_0$ which was constructed by Hoffman in [1]. We refer to these mappings when we speak of analytic maps. In the same paper he gave examples of parts whose analytic maps are homeomorphisms and parts for which this is not the case. We will refer to the former category of parts as *homeomorphic parts* and the latter as *nonhomeomorphic parts*. In this paper we will demonstrate that there is no characterization of the homeomorphic parts similar to that of nontrivial parts. Specifically, we will prove

THEOREM 1.1. *There is no family, $\mathcal{C}$, of subsets of H such that a part P of H^∞ is a homeomorphic part if and only if P is hit by the closure in the maximal ideal space of H^∞ of some set in $\mathcal{C}$.*

We accomplish the proof by studying the closure, X, in the maximal ideal space of H^∞ of the set $N = \{1 + in\colon n \text{ an integer}\}$. For each integer k let $\sigma_k(1 + in) = 1 + i(n + k)$. Since N is an interpolating sequence each part hit by X is nontrivial and as is well known X is homeomorphic to the Stone-Čech compactification of the integers. Thus, we may consider each map σ_k as extended to be a homeomorphism of X onto X. In fact, if $h_0 \in X$ and $\{1 + in(\alpha)\}$ is a net in N converging to h_0, then $\sigma_k(h_0)$ is the limit of the net $\{1 + i(n(\alpha) + k)\}$. It is not hard to see from [1] (or [2]) that if P is the part of h_0, then $X \cap P$ consists exactly of the points $\{\sigma_k(h_0)\colon k \text{ an integer}\}$. Furthermore, in terms of the analytic map, τ_{h_0}, $\sigma_k(h_0) = \tau_{h_0}(1 + ik)$.

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Hoffman proved that the closure of any sequence which tends rapidly enough to the boundary hits only homeomorphic parts. Furthermore, he demonstrated that any closed subset K of $X - N$ which is invariant under σ_1 and minimal with respect to this property intersects only nonhomeomorphic parts. It follows that both kinds of parts are represented in X.

2. Proof of the main theorem. Besides the results cited in the first section we will need the following characterization of those points of X with homeomorphic parts. If $S \subset N$ and $k \neq 0$ is an integer we call the set $\sigma_k^{-1}(S) \cap S$ the kth *level set* of S.

Lemma 2.1. *Let $h_0 \in X$. The analytic map τ_{h_0} is a homeomorphism if and only if one of the following statements hold.*

(1) *h_0 does not belong to the closure of $\{\sigma_k(h_0): k \neq 0\}$.*

(2) *There is an $S \subset N$ such that $S^- \cap \{\sigma_k(h_0): k \text{ an integer}\} = \{h_0\}$.*

(3) *There is a Blaschke product B and a constant λ of modulus 1 such that for all $z \in H$, $[(\lambda B) \circ \tau_{h_0}](z) = z$.*

(4) *Let $\{1 + in(\alpha)\}$ be a net in N converging to h_0. Then, there is an $S \subset N$ such that $h_0 \in S^-$ while $\{1 + in(\alpha)\}$ is eventually out of the kth level set of S for each k $(\neq 0)$.*

Proof. If τ_{h_0} is a homeomorphism, then clearly (1) holds. If (1) holds, then, because the sets S^-, $S \subset N$, form a basis for the topology of X, there is a set $S \subset N$ such that $h_0 \in S^-$ but $S^- \cap \{\sigma_k(h_0): k \neq 0\} = \varnothing$ and (2) follows.

To obtain (3) from (2) we recall that it was calculated in [2] that if B_0 is the Blaschke product with zeros $\{1 + in: n \text{ an integer}\}$, then (except for a possible constant factor of modulus 1) $B_0 \circ \tau_{h_0}$ is the Blaschke product with zeros $\{\tau_{h_0}^{-1}(\sigma_k(h_0)): k \text{ an integer}\}$. In fact, more was proved, namely, these zeros accumulate at exactly one point on the boundary of H where the nontangential limit values of the product are bounded away from zero. Thus, if $S \subset N$ and B is the Blaschke product with zeros S it follows that $B \circ \tau_{h_0}$ is the Blaschke product with zeros $\{\tau_{h_0}^{-1}(\sigma_k(h_0)): \sigma_k(h_0) \in S^-\}$. In the present case this latter set consists of $1 = \tau_{h_0}^{-1}(h_0)$ alone. Thus, (3) follows and, then, clearly implies that τ_{h_0} is a homeomorphism.

Finally, it is easy to see that a net $\{1 + in(\alpha)\}$ tending to h_0 is eventually in the kth level set of S if and only if $\sigma_k(h_0) \in S^-$. thus, (4) is a restatement of (2).

Examples of subsets $S \subset N$ with only homeomorphic parts represented in S^- are easily provided by subsequences $\{1 + in_k\}$, $n_k > 0$, with $n_{k+1} - n_k \to \infty$. We will call such a sequence a *fast sequence*. Let S be a sequence in N and let $k = 1, 2, \ldots$. A finite collection of successive (in the obvious sense) points of S with (Euclidean) distance k between successive points we call a *k-cluster* of S. We call the number of points in a *k-cluster* the *length* of the cluster. Using this terminology we next show how to locate points of X with nonhomeomorphic parts.

Lemma 2.2. *Let $S \subset N$. Suppose for some $k = 1, 2, \ldots$ there is a subset $S_k \subset S$ which can be written as a sequence of k-clusters whose lengths tend to infinity as one proceeds upwards in H. Then, S_k^- contains points with nonhomeomorphic parts.*

PROOF. Suppose we have proved the result for $k = 1$. Clearly $S_k \cup \sigma_1(S_k) \cup \cdots \cup \sigma_{k-1}(S_k)$ has 1-clusters satisfying the hypotheses of the lemma. It would follow that there is a point h in the closure of this set with a nonhomeomorphic part. Then, h belongs to the closure of one of the sets in the union. But the closures of sets in the union all intersect exactly the same parts. Hence, $h \in S_k^-$ as required.

It remains to prove the result for $k = 1$. Let S_0 consist of the collection of bottom-most points of the 1-clusters of S_1. Then, for each m, eventually $\sigma_m(S_0) \subset S_1$. Thus, there are parts P such that $\bar{S}_1 \cap P$ is an infinite set. This remark shows that S_1 is not the finite union of fast sequences.

Thus, there are universal nets in S_1 which are eventually out of every fast sequence. The collection, M, of limits of such nets which tend upwards in H to ∞ is closed by the iterated limit theorem. Suppose $h \in M$ but $\sigma_1(h) \notin M$. Then, the net tending to h must eventually have been in the set of uppermost points of the 1-clusters of S_1. But that set is fast. This contradiction shows that M is invariant under σ_1. Thus, M contains a closed set invariant under σ_1 which is minimal with respect to this property. Consequently, M and S^- contain points with nonhomeomorphic parts.

Using Lemma 2.2 we now prove the result from which Theorem 1.1 will easily follow.

LEMMA 2.3. *The closure of the set of points in X with nonhomeomorphic parts contains a point with homeomorphic part.*

PROOF. Construct a subset $S \subset N$ starting with 1 and proceeding upward so that the (Euclidean) distance between successive terms follows the pattern:

$$112 \quad 111223 \quad 1111222334 \quad 111112222333445\ldots.$$

Then, S is the disjoint union of interlaced sets S_k, $k = 1, 2, \ldots,$ where S_k is a sequence of k-clusters with lengths tending to infinity as in Lemma 2.2. By that lemma we may choose a net eventually in S_k which tends to a point h_k with nonhomeomorphic part. Let h be the limit of a subnet of $\{h_k\}$. Then, the "product net", $\mathfrak{N}$, constructed in the iterated limit theorem is a net in S converging to h and $\mathfrak{N}$ is eventually out of each set S_k. But it is clear that the kth level set of S is a subset of $S_1 \cup \ldots \cup S_{|k|}$. Thus, $\mathfrak{N}$ is eventually out of each level set of S so by Lemma 2.1, h has a homeomorphic part and the result follows.

PROOF OF THEOREM 1.1. Suppose $\mathcal{C}$ were a class of subsets of H such that h has homeomorphic part if and only if $h \in S^-$ for some $S \in \mathcal{C}$. Let $h \in X$ be as constructed in Lemma 2.3. Let S' be a set in $\mathcal{C}$ with h in the closure of S'. Since h is in a nontrivial part, for example, the $1/2$ pseudohyperbolic neighborhood of S' must intersect N in a sequence S with $h \in S^-$. Since S^- is open in X, it must also contain points with nonhomeomorphic parts. But every part in S^- is also a part in the closure of S'. Hence, we would have the contradiction that S' has points with nonhomeomorphic parts in its closure. Therefore, such a family $\mathcal{C}$ cannot exist.

References

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