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DENSITY OF THE RANGE OF POTENTIAL OPERATORS

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ABSTRACT. Let L be a selfadjoint operator with a closed range in a Hilbert space H and let ψ be a differentiable convex function on H. Under a nonresonance assumption, we prove that the range of $L + \partial\psi$ is dense in H.

Introduction. Let H be a real Hilbert space, let L: $D(L) \subset H \to H$ be a selfadjoint operator with a closed range and let ψ: $H \to \mathbf{R}$ be a twice Gâteaux-differentiable convex function. In [9] Mawhin showed that if there exist real numbers α, β and γ such that $0 < \beta \leqslant \gamma < \alpha$, $\sigma(L) \cap \,]-\alpha, 0[\, = \varnothing$ and, for every $u \in H$,

$$\beta I \leqslant \psi''(u) \leqslant \gamma I,$$

then $L + \partial\psi$ is one-to-one and onto. The following weaker condition as introduced by Dolph [5] in his study of Hammerstein equations

$$0 < \varliminf_{|u| \to \infty} \frac{\psi(u)}{|u|^2} \leqslant \varlimsup_{|u| \to \infty} \frac{\psi(u)}{|u|^2} < \frac{\alpha}{2}. \tag{0}$$

If (0) is satisfied, under the supplementary assumption that the right inverse of L is compact, $L + \partial\psi$ is onto (see [4] which extends some results of [2]). In the present paper we prove that (0) implies that the range of $L + \partial\psi$ is dense in H. We use the dual least action principle of Clarke and Ekeland [3] and the variational principle of Ekeland [6]. The abstract result is applied to periodic solutions of a nonlinear wave equation with a nonmonotone nonlinearity.

1. A density theorem. Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and corresponding norms $|\cdot|$. Let L: $D(L) \subset H \to H$ be a selfadjoint operator with a closed range and let ψ: $H \to \mathbf{R}$ be a differentiable convex function.

Let α, β, γ and c be real numbers such that $0 < \beta \leqslant \gamma < \alpha$ and

(A_1) $\sigma(L) \cap \,]-\alpha, 0[\, = \varnothing$, where $\sigma(L)$ denotes the spectrum of L,

(A_2) for every $u \in H$,

$$\beta \frac{|u|^2}{2} - c \leqslant \psi(u) \leqslant \gamma \frac{|u|^2}{2} + c.$$

Let us write

$$K = (L|D(L) \cap R(L))^{-1},$$
$$\psi^*(v) = \sup_{u \in H} \left[(v, u) - \psi(u) \right], \qquad v \in H,$$

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and

$$\varphi(v) = \tfrac{1}{2}(Kv, v) + \psi^*(v), \qquad v \in R(L).$$

The function ψ^* is the Fenchel transform of ψ.

The present formulation of the "dual action" φ was introduced in [1] for hyperbolic problems and in [7] for hamiltonian systems. See [4] and [8] for other abstract formulations. The following lemma has been widely used in the study of hamiltonian systems (see [7]).

LEMMA. *Under assumptions* (A_1) *and* (A_2), φ *is coercive on* $R(L)$, *i.e.* $\varphi(v) \to \infty$, $|v| \to \infty$.

PROOF. It suffices to observe that (A_1) and (A_2) imply that

$$\forall v \in R(L) \qquad -\frac{1}{\alpha}|v|^2 \leqslant (Kv, v)$$

and

$$\forall v \in H \qquad \frac{1}{\gamma}\frac{|v|^2}{2} - c \leqslant \psi^*(v). \quad \square$$

THEOREM 1. *Under assumptions* (A_1) *and* (A_2), *if* $\partial\psi$ *is uniformly continuous, the range of* $L + \partial\psi$ *is dense in* H.

PROOF. Since, for every $f \in H$, the function $\psi(u) - (f, u)$ has the same properties as $\psi(u)$, it suffices to prove that $0 \in \overline{R(L + \partial\psi)}$.

Let $\varepsilon > 0$ be fixed. By assumption there exists $\delta > 0$ such that, for every $u, v \in H$,

$$|u - v| \leqslant \delta \Rightarrow |\partial\psi(u) - \partial\psi(v)| \leqslant \varepsilon.$$

Since φ is coercive by the lemma, it follows from a theorem by Ekeland [6, p. 444] that there exists $v \in R(L)$ such that, for every $h \in R(L)$ and for every $t > 0$,

$$\varphi(v) \leqslant \varphi(v + th) + \delta t|h|.$$

Thus

$$-(Kv, h) \leqslant \frac{\psi^*(v + th) - \psi^*(v)}{t} + \delta|h| + \frac{t}{2}(Kh, h).$$

If $t\downarrow 0$, we obtain

$$-(Kv, h) \leqslant \delta^+\psi^*(v, h) + \delta|h|.$$

Since $\delta^+\psi^*(v, \cdot) + \delta|\cdot|$ is positively homogeneous and subadditive, the Hahn-Banach theorem insures the existence of $w \in \operatorname{Ker} L$ such that, for every $h \in H$,

$$(w, h) - (Kv, h) \leqslant \delta^+\psi^*(v, h) + \delta|h|.$$

But then

$$(1) \qquad -\delta|h| \leqslant \psi^*(v + h) - \psi^*(v) - (u, h)$$

where $u = w - Kv$. We shall now use a classical argument in convex analysis. It follows from (1) that the convex sets

$$C_1 = \{(h, s) \in H \times \mathbf{R}: s > \psi^*(v + h) - \psi^*(v) - (u, h)\},$$
$$C_2 = \{(h, s) \in H \times \mathbf{R}: s < -\delta|h|\}$$

are disjoint. Since C_2 is open there exists a (nonvertical) closed hyperplane separating C_1 and C_2. It is then easy to verify that there exists $f \in H$ such that, for every $h \in H$,

$$-\delta|h| \leq (f, h) \leq \psi^*(v + h) - \psi^*(v) - (u, h).$$

The first inequality implies that $|f| \leq \delta$, the second that $(u + f) \in \partial\psi^*(v)$ or $v = \partial\psi(u + f)$. By the definition of u, $Lu + \partial\psi(u + f) = 0$. Since $|f| \leq \delta$,

$$|Lu + \partial\psi(u)| = |\partial\psi(u) - \partial\psi(u + f)| \leq \varepsilon. \quad \square$$

REMARK. Particular cases of Theorem 1 were announced in **[11]** and **[12]**. The use of the "dual action" φ was suggested to us by J. L. Lions.

2. Periodic solution of a nonlinear wave equation. This section is devoted to the existence of 2π-periodic solutions in t and x of the nonlinear wave equation

$$u_{tt} - u_{xx} - u + \partial j(u) = f(t, x)$$

where $j: \mathbf{R} \to \mathbf{R}$ is convex and differentiable and $f \in H = L^2([0, 2\pi]^2)$.

Let $\mathcal{Q}$ be the linear operator defined by

$$D(\mathcal{Q}) = \big\{u \in C^2\big([0, 2\pi]^2\big): u(0, \cdot) - u(2\pi, \cdot) = u(\cdot, 0) - u(\cdot, 2\pi)$$
$$= u_t(0, \cdot) - u_t(2\pi, \cdot) = u_x(\cdot, 0) - u_x(\cdot, 2\pi) = 0\big\},$$
$$\mathcal{Q}u = u_{tt} - u_{xx}.$$

Let us write $A = \mathcal{Q}^*$. Then A is selfadjoint and $\sigma(A) = 2\mathbf{Z} + 1 \cup 4\mathbf{Z}$ consists of eigenvalues which are of finite multiplicity except 0 (see **[10]**).

Let us define $\psi: H \to \overline{\mathbf{R}}$ by

$$\psi(u) = \int_0^{2\pi}\int_0^{2\pi} j(u(t, x))\, dt\, dx.$$

THEOREM 2. *Assume that there exist $\beta, \gamma, c \in \mathbf{R}$ such that $0 < \beta \leq \gamma < 1$ and, for every $u \in \mathbf{R}$,*

$$\beta\frac{u^2}{2} - c \leq j(u) \leq \gamma\frac{u^2}{2} + c,$$

assume further that ∂j is Lipschitzian, then

$$Au - u + \partial\psi(u) = f \tag{2}$$

is solvable for f in a dense subset of H.

PROOF. It suffices to apply Theorem 1 with $L = A - I$ and $\alpha = 1$. $\square$

REMARK. Theorem 2 applies for example to

$$u_{tt} - u_{xx} - \tfrac{1}{4}u + \tfrac{3}{4}\sin u = f(t, x).$$

In this case

$$\partial j(u) = \tfrac{3}{4}(u + \sin u).$$

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