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A CHARACTERIZATION OF COMPLETE METRIC SPACES

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ABSTRACT. A general formulation of the completeness argument used in the Bishop-Phelps Theorem and many other places has been given by Ekeland. It is shown that Ekeland's formulation characterizes complete metric spaces.

A central idea in the proof of the Bishop-Phelps Theorem is the use of norm completeness and a partial ordering to produce a point where a linear functional attains its supremum on a closed bounded convex set. In fact, this completeness technique is useful in many situations as has been described in the surveys of Phelps [7] and Brezis-Browder [1]. Recently Ekeland [3] has given a very general formulation of this technique and has applied it to a wide variety of problems [4]. In the present note we show that Ekeland's formulation is actually equivalent to completeness for metric spaces.

Ekeland's Theorem may be stated as follows:

THEOREM 1. *Let (M, d) be a complete metric space, and $F\colon M \to \mathbf{R} \cup \{+\infty\}$ a lower semicontinuous function, $F \not\equiv +\infty$, bounded from below. Let $\varepsilon > 0$ be given and a point $u \in M$ such that*

$$F(u) \leqslant \inf_M F + \varepsilon.$$

Then there exists a point $v \in M$ such that

(i) $F(v) \leqslant F(u)$.

(ii) $d(u, v) \leqslant 1$.

(iii) *For all $w \neq v$, $F(w) + \varepsilon d(v, w) > F(v)$.*

This result characterizes completeness of M in the following sense.

THEOREM 2. *Let (M, d) be a metric space. Then M is complete if and only if for every continuous $F\colon M \to \mathbf{R} \cup \{+\infty\}$, $F \not\equiv +\infty$, bounded from below and every $\varepsilon > 0$ there is a point $v \in M$ satisfying*

(i)′ $F(v) \leqslant \inf_M F + \varepsilon$ *and* (iii) *above.*

PROOF. The "only if" direction is immediate from Theorem 1. For the converse assume that (M, d) is an arbitrary metric space satisfying the hypotheses. Let $(y_n) \subseteq M$ be a Cauchy sequence and consider the function $F\colon M \to \mathbf{R}$ given by

$$F(x) = \lim_n d(y_n, x).$$

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The function F is continuous and, since (y_n) is Cauchy, $\inf F = 0$. To show completeness we produce a $v \in M$ such that $F(v) = 0$.

Choose any $0 < \varepsilon < 1$. Now, from (i)′ and (iii), there is a $v \in M$ with $F(v) \leqslant \varepsilon$ and $F(w) + \varepsilon d(w, v) > F(v)$ for all $w \neq v$. From the definition of F and the fact that (y_n) is Cauchy we can take $w = y_p$ for p large such that $F(w)$ is arbitrarily small and $d(w, v) \leqslant \varepsilon + \eta$ for any $\eta > 0$, because $F(v) \leqslant \varepsilon$. Using condition (iii) we get that, in fact, $F(v) \leqslant \varepsilon^2$. Repeating the argument we may conclude that $F(v) \leqslant \varepsilon^n$ for all $n \geqslant 1$ and so $F(v) = 0$ as required. Q.E.D.

REMARK. Caristi [2] and Kirk [5] have studied connections between completeness and fixed-point theorems. In [6] Kirk proves that a fixed point property related to Ekeland's formulation also characterizes complete metric spaces.

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