NON-(CA) ANALYTIC GROUPS AND GROUPS OF ROTATIONS

T. CHRISTINE STEVENS

Abstract. It is known that every non-(CA) analytic group arises from the action of a vector group on a (CA) analytic group. We prove here that this action always involves a vector group of rotations acting on another vector group.

1. Introduction. For any analytic group $G$, the group $\text{Aut}(G)$ of all continuous automorphisms of $G$ is a Lie group in the generalized compact-open topology. (See, for example, p. 40 of [2].) $G$ is a (CA) analytic group if the group of inner automorphisms of $G$ is closed in $\text{Aut}(G)$.

Zerling [7, p. 182] has shown that every non-(CA) analytic group arises from the action of a vector group on a (CA) analytic group. The examples in [3], [4], [6], and [7], however, take a much less general form: all involve the action of a vector group of rotations on another vector group. In the present article we show that this is an essential feature of all non-(CA) analytic groups, rather than an accidental property of the particular examples in the literature. Although we will confine our attention to simply-connected, solvable analytic groups, this restriction does not materially affect the scope of our result. For an analytic group $G$ is (CA) if and only if its simply-connected covering group $G'$ is (CA), since $G$ is the quotient of $G'$ by a central subgroup. Moreover, Van Est [5, pp. 559–561] has demonstrated that $G$ is (CA) if and only if its radical is (CA).

Before stating our result precisely, we must introduce some notation. First we describe in more detail the principal result in [7]. If $G$ is a non-(CA) analytic group, then $G$ has the form $M \otimes V$, where $M$ is a (CA) analytic group and $V$ is a vector subgroup of $\text{Aut}(M)$ whose closure in $\text{Aut}(M)$ is compact. We will say that $G = M \otimes V$ is a standard decomposition of $G$.

Now let $\theta$ be an element of the set $R$ of real numbers. We will let $R_\theta$ denote the 2-by-2 matrix which corresponds to the rotation of $R^2$ through a counterclockwise angle of $\theta$. If $n$ is an even natural number, let $T_n$ be the subgroup of $\text{Gl}(n, R)$ consisting of all matrices of the form shown in Figure 1, where $k = n/2$ and $\theta_1, \ldots, \theta_k \in R$. If $n > 1$ is odd, we define a homomorphism $h: \text{Gl}(n - 1, R) \to \text{Gl}(n, R)$ by

\[h(A) = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}, \quad A \in \text{Gl}(n - 1, R),\]

and then let $T_n = h(T_{n-1})$. We can now proceed to our theorem.
THEOREM. Let $G$ be a simply-connected, solvable, non-(CA) analytic group with standard decomposition $G = M \circledast V$. Let $G_1$ and $G_2$ be, respectively, the derived groups of $G$ and $G_1$. Then $G_1 \subseteq M$ and $(G_1/G_2) \circledast V$ is a non-(CA) analytic group which is isomorphic, both topologically and algebraically, with $\mathbb{R}^n \circledast W$, where $n > 4$ is the dimension of $G_1/G_2$ and $W$ is a vector subgroup of $T^n$ whose closure in $T^n$ is compact.

2. Proof of theorem. Since $V$ is abelian and $G$ is not, it is clear that $G_1 \subseteq M$ and $G_1 \neq G_2$. That $G_1$ and $G_1/G_2$ are simply-connected analytic groups follows from the fact that $G$ is simply-connected and $G_1$ and $G_2$ are normal in $G$ [2, pp. 135–136]. Since $G_1/G_2$ is also abelian, it follows that $G_1/G_2$ is isomorphic to $\mathbb{R}^n$ for some $n > 0$.

We claim that $V$ acts effectively on the vector group $G_1/G_2$. From Theorem 3.2 of [7] we know that $V$ acts effectively on $G_1$ and may thus be regarded as a vector subgroup of $\text{Aut}(G_1)$ with compact closure $S$. To establish our claim, we will show that $S$ acts effectively on $G_1/G_2$. Let $g_1$ and $g_2$ be the Lie algebras of $G_1$ and $G_2$, respectively. The isomorphism of $\text{Aut}(G_1)$ with $\text{Aut}(g_1)$ defines a representation of $S$ into $\text{Gl}(g_1)$ which must be semisimple because $S$ is compact. Therefore there exists an $S$-invariant subspace (not necessarily a subalgebra) $\mathfrak{m}$ of $g_1$ such that $g_1 = g_2 \oplus \mathfrak{m}$. If $s \in S$ induces the identity automorphism on $G_1/G_2$, then $S$ acts as the identity not only on $\mathfrak{m}$ but also on the Lie algebra $\mathfrak{a}$ generated by $\mathfrak{m}$. An exercise in [1, Exercise 4, p. 91], however, shows that $\mathfrak{a} = g_1$, so that $s$ must be the identity on all of $G_1$. This demonstrates that $S$ acts effectively on $G_1/G_2$ (and, incidentally, that $n > 2$).

To finish the proof, let us choose some specific isomorphism of $G_1/G_2$ with $\mathbb{R}^n$. Regarding $V$ as a nonclosed vector subgroup of $\text{Gl}(n, \mathbb{R})$, we know that $V$ is contained in a maximal torus $T$ of $\text{Gl}(n, \mathbb{R})$. Examination of the centralizer of $T_n$ in $\text{Gl}(n, \mathbb{R})$ reveals that $T_n$ is also a maximal torus in $\text{Gl}(n, \mathbb{R})$ and is therefore conjugate to $T$ [2, pp. 152, 181]. Let $A$ be an element of $\text{Gl}(n, \mathbb{R})$ such that
$T_n = AT_A^{-1}$, and let $W = AV_A^{-1}$. It is easy to verify that the function $f: (G_1/G_2) \otimes V \to \mathbb{R}^n \otimes W$ defined by $f(x, v) = (A(x), AvA^{-1})$ for $x \in G_1/G_2, v \in V$, is an isomorphism of Lie groups. Since $W$ is a nonclosed vector subgroup of $T_n$, it follows immediately that $n \geq 4$ and that neither $\mathbb{R}^n \otimes W$ nor $(G_1/G_2) \otimes V$ is $(CA)$.

**References**