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SUBGROUPS OF COMPACT LIE GROUPS CONTAINING A MAXIMAL TORUS ARE CLOSED

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ABSTRACT. We prove the claim made in the title. As a corollary, we obtain that a compact Lie group G has only finitely many subgroups containing a fixed maximal torus. The special case $G = U(n)$ was dealt with in a recent paper of Borevich and Krupeckiĭ.

Introduction. We denote by G a compact Lie group and by T a maximal torus of G. The Lie algebras of Lie groups will be denoted by corresponding German letters. Thus $\mathfrak{g}$ (resp. $\mathfrak{t}$) is the Lie algebra of G (resp. T). The connected closed subgroups H of G such that $H \supset T$ have been studied and classified by Borel and Siebenthal **[1]**. It is known that every connected Lie subgroup H of G containing T is closed in G. For this see, for instance, **[4**, Theorem 8.10.1**]** or Lemma 2 below. Our main result (Theorem 3) is that *every* subgroup H of G containing T is necessarily closed in G.

This note is motivated by a recent paper of Borevich and Krupeckiĭ **[2]** where they study the case $G = U(n)$. They take T to be the group of diagonal matrices in $U(n)$ and describe all subgroups H of $U(n)$ containing T. Our Theorem 3 together with Borel and Siebenthal classification of connected closed subgroups containing T can be viewed as a partial generalization of **[2]** to arbitrary compact Lie groups.

By $N(H)$ we denote the normalizer of a subgroup H in G. We set $N(T) = N$. Similarly, if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ then $\mathfrak{n}(\mathfrak{h})$ is its normalizer in $\mathfrak{g}$, and $N(\mathfrak{h})$ its normalizer in G. If H is a closed subgroup of G then H^0 denotes its identity component. If H is any subgroup of G then $\bar{H}$ denotes its closure.

Results and proofs. We start with a simple lemma about Lie algebras of compact groups.

LEMMA 1. *Let* $\mathfrak{h}$ *be a subalgebra of* $\mathfrak{g}$ *containing* $\mathfrak{t}$. *Then* $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$.

PROOF. Let $\langle X, Y\rangle$ be an invariant inner product on $\mathfrak{g}$. Assume that there exists $X \in \mathfrak{n}(\mathfrak{h})$ such that $X \notin \mathfrak{h}$. Then $\mathfrak{h}$ is an ideal of $\mathfrak{h}_1 = \mathfrak{h} + \mathbf{R}X$. Let $\mathbf{R}Y$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{h}_1$. Then $\mathbf{R}Y$ is also an ideal of $\mathfrak{h}_1$. Hence $\mathfrak{t} + \mathbf{R}Y$ is abelian. This is impossible since $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$.

LEMMA 2. *If H is a Lie subgroup of G and $H \supset T$ then*
(i) $H^0 = N(H^0)^0$ *and* $\bar{H} = H$;
(ii) $H = H^0 \cdot (H \cap N)$.

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PROOF. (i) $N(H^0)$ is closed in G and its Lie algebra is $\mathfrak{n}(\mathfrak{h})$. By Lemma 1, $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and so $H^0 = N(H^0)^0$. Since $N(H^0)$ is compact and $H^0 \subset H \subset N(H^0)$, it follows now that H is closed in G.

(ii) For $x \in H$ we have $xTx^{-1} \subset H^0$. There exists $y \in H^0$ such that $yTy^{-1} = xTx^{-1}$. Hence $x = yz$ where $z \in H \cap N$.

THEOREM 3. *let G be a compact Lie group, T a maximal torus of G, and H any subgroup of G containing T. Then H is closed in G.*

PROOF. Let K be the subgroup of G generated by all conjugates xTx^{-1} with $x \in H$. Since $T \subset H$, we have $K \subset H$. Clearly, K is arcwise connected, and so it is a connected Lie subgroup of G, see **[3]**. Since $K \supset T$, by Lemma 2, we have $K = N(K)^0 \subset H \subset N(K)$. Hence H is closed in G.

COROLLARY. *There are only finitely many subgroups of G containing T.*

PROOF. Let H be such a subgroup. By Theorem 3 H is closed in G. By Lemma 2 we have $H = H^0 \cdot (H \cap N)$. There are only finitely many choices for H^0, see **[1]**. Since $T = N^0 \subset H \cap N \subset N$, there are also only finitely many choices for $H \cap N$.

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