BANACH SPACES WHICH ALWAYS CONTAIN SUPREMUM-ATTAINING ELEMENTS

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ABSTRACT. It is proved that if a weakly compactly generated Banach space $X$ has the property that, for every closed, bounded convex subset $K$ of $X^*$, there exists a nonzero element of $X$ which attains its supremum on $K$, then $X$ contains no copy of $l^1$.

Let $X$ be a Banach space and let $A$ be a bounded subset of its dual. Define

$$\mathcal{SA}(A) = \{ x \in X \setminus \{0\} : \hat{x} \text{ attains its supremum on } A \}.$$  

Here $\hat{x}$ denotes the image of $x$ under the natural embedding of $X$ in $X^{**}$. Also define

$$\mathcal{SEX}(A) = \{ x \in X : \hat{x} \text{ strongly exposes } A \}.$$  

Here, a functional $f$ strongly exposes a set $B$ if the diameter of the slice

$$S(B, f, \alpha) = \{ b \in B : f(b) > \alpha \}$$

approaches 0 as $\alpha$ approaches $\max f(B)$ from below.

In case $A \subset X^*$ is norm-closed we have $\mathcal{SEX}(A) \subset \mathcal{SA}(A)$. This inclusion is usually proper.

The present work was motivated by the following results.

**Theorem A.** The dual space $X^*$ has the Radon-Nikodým property if and only if $\mathcal{SEX}(A) \neq 0$ for all norm-closed, bounded, convex subsets, $A$ of $X^*$.

**Theorem B.** The dual space $X^*$ has the Radon-Nikodým property if and only if $\mathcal{SA}(A)$ is of 2nd category for all norm-closed, bounded, convex subsets $A$ of $X^*$.

The reader should refer to [1] or [2] for information on the Radon-Nikodým property (including its definition). Theorem A follows from a result of Namioka and Phelps [4] combined with one of Stegall [7]. Indeed, their results imply that, if $X^*$ has the Radon-Nikodým property and $A$ is a norm-closed, convex subset, then $\mathcal{SEX}(A)$ is a dense $G_δ$ subset of $X$. This fact also proves half of Theorem B. The other half is a very easy modification of an argument due to Bourgain (see [1], [6]).

Theorems A and B, together with a spirit of optimism, led to

**Conjecture.** If $\mathcal{SA}(A) \neq \emptyset$ for all norm-closed, bounded, convex subsets $A$ of $X^*$, then $X^*$ has the Radon-Nikodým property.

Let us refer to the property of $X$ which is conjectured to imply the RNP in $X^*$ as Property $SA^*$. Our result is weaker than the one conjectured. We were forced to
assume that $X$ is weakly compactly generated. This allows a reduction to the separable case. The desired conclusion is that every separable subspace of $X$ has separable dual (that this implies that $X^*$ has the RNP is due to Uhl [8] generalizing Dunford and Pettis [3]). We obtain the weaker conclusion that $l^1$ does not embed in $X$.

A lemma is needed.

**Lemma.** $C[0, 1]$ does not have property $SA^*$.

**Proof.** Let $(x_n)$ be a dense sequence in $[0, 1]$. For each $n = 1, 2, \ldots$, let $\mu_n$ be the point mass at $x_n$ with total mass $n/(n + 1)$. Let

$$A = \overline{\text{co}}\left((\mu_n) \cup (-\mu_n)\right) \subseteq C([0, 1])^*.$$

It is easy to see that

$$(1) \quad \sup\{\langle \mu, f \rangle : \mu \in A\} = \|f\|,$$

for any $f \in C([0, 1])$. Now we show that $\mu \in A$ then $\mu$ can be expressed as

$$(2) \quad \mu = \sum_{n=1}^{\infty} a_n \mu_n,$$

where $\sum_{1}^{\infty} |a_n| < 1$. To prove this, let $T$ be the map from $l^1$ into $C([0, 1])^*$ which takes $(a_n) \in l^1$ to $\sum_{1}^{\infty} a_n \mu_n \in C([0, 1])^*$. Then $T$ is clearly an isomorphism. Since $A$ is obviously the image, under $T$, of the unit ball of $l^1$, (2) holds.

To finish the proof of the Lemma, we will show that no nonzero $f$ in $C([0, 1])$ attains its supremum on $A$. Suppose the contrary. Then, using (1), there exists $f$ in $C([0, 1])$ and $\mu \in A$ such that

$$\langle \mu, f \rangle = \|f\| > 0.$$

Expressing $\mu$ as in (2), we have

$$\|f\| = \langle \mu, f \rangle = \sum_{1}^{\infty} a_n \langle \mu_n, f \rangle < \sum_{1}^{\infty} |a_n| |\langle \mu_n, f \rangle|\
\quad < \sum_{1}^{\infty} |a_n| \|f\| < \|f\|.$$

This contradiction completes the proof of the Lemma.

**Theorem.** Let $X$ be a weakly compactly generated Banach space with property $SA^*$. Then $l^1$ does not embed in $X$.

**Proof.** We first observe that every quotient of a space with property $SA^*$ also has this property. For, suppose $Q$ is a bounded linear operator from a Banach space $Y$ onto a Banach space $Z$. Suppose $K$ is a closed, bounded, convex set in $Z^*$ with $SA(K) = \emptyset$. A moment's reflection shows that $SA(Q^*K) = \emptyset$ and so $Y$ fails to have $SA^*$.

Now suppose that $l^1$ embeds in $X$. Since $X$ is weakly compactly generated there is a separable subspace $Y$ of $X$ which contains a copy of $l^1$ and which is complemented in $X$. Then $Y$ also has property $SA^*$. But it follows from a result of
Pelczynski [5] that a separable space containing \( l^1 \) has \( C[0, 1] \) as a quotient. Hence \( C[0, 1] \) has \( SA^* \) and we have arrived at a contradiction. This completes the proof.

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REFERENCES

6. J. Rainwater, Univ. of Washington Seminar Notes, Fall 1976.

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