A NOTE ON INTERTWINING $M$-HYPONORMAL OPERATORS


Abstract. If $AX = XB^*$ with $A$ and $B$ $M$-hyponormal, then $A^*X = XB$. Furthermore, $(\text{ran } X)^\perp$ reduces $A$, ker $X$ reduces $B$, and $A|_{(\text{ran } X)^\perp}$ and $B^*|_{\text{ker } X}$ are unitarily equivalent normal operators. An asymptotic version is also proved.

Let $\mathcal{H}$ be a Hilbert space. A bounded operator $A$ on $\mathcal{H}$ is called dominant by J. Stampfli and B. Wadhwa [4] if, for all complex $\lambda$, range$(A - \lambda) \subseteq \text{range}(A - \lambda)^*$, or, equivalently, if there is a real number $M_\lambda > 1$ such that

$$\| (A - \lambda)^* f \| \leq M_\lambda \| (A - \lambda) f \|$$

for all $f$ in $\mathcal{H}$. If there is a constant $M$ such that $M_\lambda < M$ for all $\lambda$, $A$ is called $M$-hyponormal, and if $M = 1$, $A$ is hyponormal.

Stampfli and Wadhwa showed in [4, Theorem 1] that if $A$ is dominant, $B$ is hyponormal, $X$ is one-to-one and has dense range, and if $AX = XB^*$, then $A$ and $B$ are normal. M. Radjabalipour improved this result by allowing $B$ to be $M$-hyponormal [3, Theorem 3(a)]. Of course, the condition that $A$ and $B$ are normal allows one to conclude immediately by the usual Putnam-Fuglede theorem that $A^*X = XB$. S. K. Berberian [2] has obtained the latter result under the conditions that $A$ and $B$ are hyponormal and $X$ is Hilbert-Schmidt (but not one-to-one or with dense range). It seems to have escaped notice, however, that if $A$ and $B$ are both $M$-hyponormal, the conclusion that $A^*X = XB$ can be reached with no restrictions on $X$ at all; moreover, by employing both intertwining equations one can determine precisely the subspaces on which $A$ and $B$ must be normal. We will need two other results from [3]:

**Theorem A (Radjabalipour).** Let $A$ be dominant and let $\mathcal{M}$ be an invariant subspace of $A$ for which $A|_{\mathcal{M}}$ is normal. Then $\mathcal{M}$ reduces $A$.

**Theorem B (Radjabalipour).** If $A$ and $A^*$ are $M$-hyponormal then $A$ is normal.

We begin with a symmetric version.

**Theorem 1.** Let $A$ be $M$-hyponormal and suppose that $AX =XA^*$. Then $A^*X = XA$.

Received by the editors September 12, 1980.

1980 Mathematics Subject Classification. Primary 47B20.

Key words and phrases. Normal operator, $M$-hyponormal operator, intertwining.

The first and third authors were partially supported by grants from the Research Grants Committee of the University of Alabama.

© 1981 American Mathematical Society

0002-9939/81/0000-0516/$01.75

514
PROOF. Let $X = H + iJ$ be the Cartesian decomposition of $X$. By taking the adjoint of the intertwining equation, we obtain $AX^* = X^*A^*$ and thus $AH = HA^*$ and $AJ = JA^*$.

Let $\mathcal{M}$ be the kernel of $H$ and decompose the Hilbert space as $\mathcal{M}^\perp \oplus \mathcal{M}$. $\mathcal{M}$ is clearly invariant for $A^*$ and we can represent $A$ and $H$ as operator matrices:

$$A = \begin{pmatrix} C & D \\ 0 & E \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}. $$

$C$ is $M$-hyponormal and since $AH = HA^*$ we have $CK = KC^*$ and because $K$ is one-to-one and has dense range we conclude that $C$ is normal by [3, Theorem 3(a)]. By Theorem A, $D = 0$ and it follows that $A^*H = HA$. Similarly, $A^*J = JA$ and thus $A^*X = XA$.

**Theorem 2.** If $A$ and $B$ are $M$-hyponormal and $AX = XB^*$ then $A^*X = XB$.

**Proof.** Let

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}. $$

$\tilde{A}$ is $M$-hyponormal and $\tilde{A}\tilde{X} = \tilde{X}\tilde{A}^*$ and Theorem 1 yields the desired result.

The next theorem, which generalizes Theorem 3(a) of [3], identifies the subspaces on which $A$ and $B$ must be normal.

**Theorem 3.** Let $A$, $B$ and $X$ be as in Theorem 2. Then

(a) $(\text{ran } X)^\perp$ reduces $A$ and $\ker X$ reduces $B$.

(b) $A|(\text{ran } X)^\perp$ and $B^*|\ker X$ are unitarily equivalent normal operators.

**Proof.** (a) By Theorem 2, $AXX^* = XB^*X^* = XX^*A$. Thus $A$ commutes with $XX^*$ and so $(\text{ran } X)^\perp = (\text{ran } XX^*)^\perp$ reduces $A$. Similarly $B$ commutes with $X^*X$ and $\ker X = \ker X^*X$ reduces $B$.

(b) Let $X = UP$ be the polar decomposition of $X$. Since $B$ commutes with $P$ as above, we have

$$(AU - UB^*)P = 0.$$ 

Let $\mathcal{K}_1 = \ker^\perp X = \ker^\perp P$ and let $\mathcal{K}_2 = (\text{ran } X)^\perp$; let $A_2 = A|\mathcal{K}_2$ and $B_1 = B|\mathcal{K}_1$. Let $V: \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be defined by $Vf = Uf$ for all $f \in \mathcal{K}_1$. The equation above then becomes

$$A_2V = VB_1^*. $$

Since $V$ is an invertible isometry we have that $A_2$ is unitarily equivalent to $B_1^*$ and since $A_2$ and $B_1$ are both $M$-hyponormal, Theorem B implies that both are normal. The proof is complete.

We now proceed to an asymptotic version, which is most readily attained by employing some machinery developed by Berberian [1]. We sketch Berberian's construction here; the details are in [1]. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}$ be the set of bounded sequences of vectors $(f_n)$, with $f_n \in \mathcal{H}$. Let "glim" denote a generalized limit defined on the collection of bounded sequences of complex numbers, and let $\mathcal{K} = \{(f_n) \in \mathcal{B}: \text{glim}(\|f_n\|) = 0\}$. Then the set $\mathcal{P} = \mathcal{B}/\mathcal{K}$ has a pre-Hilbert space structure with the inner product $(\langle f_n + \mathcal{K}, g_n + \mathcal{K} \rangle) = \text{glim}(f_n, g_n)$. The map $f \rightarrow \{f, f, \ldots\} + \mathcal{K}$ is a natural imbedding of $\mathcal{H}$ into $\mathcal{P}$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Let \( \mathcal{K} \) be the completion of \( \mathcal{P} \). If \( \{ T_n \} \) is a bounded sequence of operators on \( \mathcal{K} \) and if \( \{ f_n \} \subset \mathcal{K} \), then the sequence \( \{ T_n f_n \} \subset \mathcal{K} \) and it follows that the function that maps \( \{ g_n \} \subset \mathcal{K} \) to \( \{ T_n g_n \} \subset \mathcal{K} \) defines a bounded linear operator on \( \mathcal{K} \) which we call \( \phi((T_n)) \). It is easy to check that \( \phi((T_n)) = 0 \) if and only if \( \| T_n \| \to 0 \), that \( \phi((T_n^*)) = \phi((T_n))^* \), and that \( \phi((T_n)) \) is positive if and only if \( T_n - |T_n| \to 0 \), in the strong operator topology.

**Theorem 4.** Let \( \{ T_n \} \) and \( \{ S_n \} \) be bounded sequences for which there exists a number \( M \) such that, for all complex numbers \( \lambda \),

\[
M^2(T_n - \lambda)^*(T_n - \lambda) - (T_n - \lambda)(T_n - \lambda)^* \\
- |M^2(T_n - \lambda)^*(T_n - \lambda) - (T_n - \lambda)(T_n - \lambda)^*| \to 0 \quad \text{(strongly)}
\]

and

\[
M^2(S_n - \lambda)^*(S_n - \lambda) - (S_n - \lambda)(S_n - \lambda)^* \\
- |M^2(S_n - \lambda)^*(S_n - \lambda) - (S_n - \lambda)(S_n - \lambda)^*| \to 0 \quad \text{(strongly)}.
\]

Let \( \{ X_n \} \) be a bounded sequence and suppose that \( T_n X_n - X_n^* S \to 0 \). Then \( T_n^* X_n - X_n S_n \to 0 \).

**Proof.** The conditions on \( \{ T_n \} \) and \( \{ S_n \} \) imply that \( \phi((T_n)) \) and \( \phi((S_n)) \) are \( M \)-hyponormal. The equation

\[
\phi((T_n))\phi((X_n)) = \phi((X_n))\phi((S_n))^*
\]

holds, and Theorem 2 yields the result.

**Corollary.** If \( T \) and \( S \) are \( M \)-hyponormal and if \( TX_n - X_n^* S \to 0 \) for a bounded sequence \( \{ X_n \} \), then \( T^* X_n - X_n S \to 0 \) as well.

**References**


**Department of Mathematics, University of Alabama, University, Alabama 35486**

**Department of Mathematics, Texas A&M University, College Station, Texas 77843**

**Department of Mathematics, University of Alabama, University, Alabama 35486**