SOME INTRINSIC COORDINATES ON TEICHMÜLLER SPACE

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ABSTRACT. We give a new construction of intrinsic global coordinates on the Teichmüller space $T_p$ of closed Riemann surfaces of genus $p > 2$. Our construction produces an injective holomorphic map from $T_p$ into the space of Schottky groups of genus $p$.

1. Introduction. Since the Teichmüller space $T_p$ of closed Riemann surfaces of genus $p > 2$ is a complex analytic manifold of dimension $n = 3p - 3$, any injective holomorphic map $f : T_p \rightarrow \mathbb{C}^n$ defines a set of global coordinate functions on $T_p$. We call these coordinates intrinsic if the coordinates $f(t)$ are determined from the marked Riemann surface $t$ alone and do not depend on the choice of a basepoint $t_0$ in $T_p$. In this paper we describe a new way to define intrinsic coordinates on $T_p$.

We should emphasize that we are defining complex coordinates for the complex manifold $T_p$. The problem of finding real analytic coordinates was solved classically with the help of Fuchsian groups. The first global complex coordinates were found by Bers [2], using quasi-fuchsian groups. The Bers coordinates depend on the choice of a basepoint. Maskit [5] defined the first intrinsic (complex) coordinates. Our coordinates are closer in spirit to the Bers coordinates since we use quasi-fuchsian groups. It would, of course, be interesting to find global coordinates for $T_p$ that do not depend on uniformization by Kleinian groups.

2. Quasifuchsian groups. Let $\Gamma$ be a quasifuchsian group of type $(p, 0)$. This means that the limit set $\Lambda(\Gamma)$ is a Jordan curve in the extended plane, that $\Gamma$ maps each of the Jordan regions $D_1$ and $D_2$ bounded by $\Lambda(\Gamma)$ into itself, and that the quotient maps $D_1 \rightarrow D_1/\Gamma$ and $D_2 \rightarrow D_2/\Gamma$ are unramified coverings of closed Riemann surfaces of genus $p$.

Lifting a canonical dissection of the surface $D_1/\Gamma$ to $D_1$, we can choose an ordered $2p$-tuple

$$\sigma = (A_1, B_1, A_2, B_2, \ldots, A_p, B_p)$$

of Möbius transformations such that the $A_j$ and $B_j$ generate $\Gamma$ and satisfy the relation

$$\prod_{j=1}^{p} C_j = I, \quad C_j = A_j B_j A_j^{-1} B_j^{-1}.$$
The pair \((\sigma, \Gamma)\) is called a marked quasifuchsian group. We say that \((\sigma, \Gamma)\) is normalized if the attractive fixed points of \(B_1\) and \(B_2\) and the repulsive fixed point of \(B_1\) are at 0, 1, and \(\infty\) respectively. Our reason for that normalization will become clear in §5. It is well known (see [3]) that the space of normalized groups \((\sigma, \Gamma)\) is a complex manifold, biholomorphically equivalent to \(T_p \times T_p\).

To represent \(T_p\) by a set of normalized groups one must embed \(T_p\) in \(T_p \times T_p\). The Bers coordinates are obtained by identifying \(T_p\) with a “slice” \(T_p \times \{t_0\}\) in \(T_p \times T_p\). Our method is to identify \(T_p\) with the diagonal. Our main theorem gives a general procedure for making that identification, and in §5 we illustrate how to use the main theorem to define intrinsic coordinates on \(T_p\).

3. The main theorem. By definition, if \((\sigma, \Gamma)\) is a marked quasifuchsian group the 2\(p\)-tuple \(\sigma\) induces a canonical dissection of \(D_1/\Gamma\). The induced dissection of \(D_2/\Gamma\), however, is not canonical, because of orientation. To identify the space of normalized groups with \(T_p \times T_p\) we use a sense-reversing diffeomorphism to make the dissection of \(D_2/\Gamma\) canonical. The following theorem describes the diagonal.

**Theorem 1.** Let \(W\) be a closed Riemann surface of genus \(p > 2\) with a canonical homotopy basis \(a_1, \ldots, b_p\), and let \(\theta\) be an automorphism of \(\pi_1(W)\) induced by a sense-reversing diffeomorphism of \(W\). There is a unique normalized marked quasifuchsian group \((\sigma, \Gamma)\) such that:

(i) the map from \(\pi_1(W)\) to \(\Gamma\) that sends \(a_j\) to \(A_j\) and \(b_j\) to \(B_j\), \(1 < j < p\), is induced by a conformal map from \(W\) to \(D_1/\Gamma\),

(ii) there is a conformal map \(F: D_2 \rightarrow D_1\) such that

\[
F(\gamma z) = \theta(\gamma) F(z) \quad \text{for all } \gamma \in \Gamma, z \in D_2.
\]

If \(\theta\) has order two, then \(F\) is a Möbius transformation of order two, and \(F\) and \(\Gamma\) generate a Kleinian group whose deformation space is \(T_p\).

Notice that in (ii) we use the isomorphism (i) between \(\pi_1(W)\) and \(\Gamma\) to interpret \(\theta\) as an automorphism of \(\Gamma\). We refer the reader to [3] for a discussion of deformation spaces of Kleinian groups.

4. Proof of Theorem 1. This result is really a corollary of the simultaneous uniformization theorem of Bers [1], but we find it simplest to give a direct proof, modelled on the proof of [1]. First we choose a holomorphic universal covering of \(W\) by the upper half-plane \(U\), identifying \(\pi_1(W)\) with the group \(G\) of deck transformations in the usual way. We normalize \(G\) so that the attractive fixed points of \(b_1\) and \(b_2\) and the repulsive fixed point of \(b_1\) are at 0, 1, and \(\infty\) respectively. By hypothesis there is a sense-reversing diffeomorphism of \(W\) that induces the automorphism \(\theta\). Lifting to \(U\) we get a diffeomorphism \(f: U \rightarrow U\) such that \(f(gz) = \theta(g)f(z)\) for all \(g \in G, z \in U\). Put \(h(z) = f(z)\) for \(z\) in the lower half-plane \(U^*\). Then \(h\) is a sense-preserving diffeomorphism of \(U^*\) onto \(U\), and

\[
h(gz) = \theta(g)h(z) \quad \text{for all } g \in G, z \in U^*.
\]

Now let \(w: \mathbb{C} \rightarrow \mathbb{C}\) be a quasiconformal map such that \(w\) fixes the points 0, 1, and \(\infty\), and both \(w\) and \(w \circ h^{-1}\) are conformal in \(U\) (i.e., \(w \circ h^{-1}\) is a conformal map in \(U\), and...
$w_2/w_z = h_2/h_z$ in $U^*$. Put $\Gamma = wGW^{-1}$, define the isomorphism $\varphi: G \to \Gamma$ by

$$\varphi(g) = wgw^{-1} \text{ for all } g \in G,$$

and put $\sigma = (A_1, B_1, \ldots, A_p, B_p)$, $A_j = \varphi(a_j)$, $B_j = \varphi(b_j)$, $1 < j < p$. Then $(\sigma, \Gamma)$ is a normalized marked quasifuchsian group with $D_1 = w(U)$ and $D_2 = w(U^*)$, and the conformal map $w: U \to D_1$ induces a conformal map of $W$ onto $D_1/\Gamma$ that satisfies (i). Moreover $F = wwh^{-1}: D_2 \to D_1$ is conformal, and (3) and (4) give

$$F\varphi(g)F^{-1} = whgh^{-1}w^{-1} = w\varphi(\varphi(g))$$

in $D_1$, so $(\sigma, \Gamma)$ satisfies (i) and (ii).

Suppose $(\sigma', \Gamma')$ is another normalized group that satisfies (i) and (ii) with $F'$: $D_2' \to D_1'$ conformal. Write $\sigma' = (A'_1, \ldots, B'_p)$. Then (i) gives a conformal map $H: D_1 \to D_1'$, so that in $D_1'$ we have

$$HA_jH^{-1} = A'_j, \quad HB_jH^{-1} = B'_j, \quad 1 < j < p.$$

Put $C = H$ in $D_1$ and $C = (F')^{-1}HF$ in $D_2$. Then $C$ maps the regular set of $\Gamma$ conformally onto the regular set of $\Gamma'$ and induces an isomorphism of $\Gamma$ onto $\Gamma'$, so the Marden isomorphism theorem [4] implies that $C$ is a Möbius transformation. The normalization implies that $C$ is the identity, so $(\sigma, \Gamma) = (\sigma', \Gamma')$ and $(\sigma, \Gamma)$ is unique.

Finally, let $\theta$ have order two. Put $C = F$ in $D_2$ and $C = F^{-1}$ in $D_1$. Then $C\gamma C^{-1} = \theta(\gamma)$ in both $D_1$ and $D_2$, so the Marden isomorphism theorem again implies that $C$ is a Möbius transformation. By construction $C$ has order two. Since $F = C$ in $D_2$, $F$ is itself (extendible to) a Möbius transformation of order two. It is clear from (ii) that the group $H$ generated by $\Gamma$ and $F$ is Kleinian, and $\Gamma$ is the subgroup of index two that maps the region $D_1$ onto itself. By §7 of [3], the deformation space of $H$ is biholomorphically equivalent to $T_p$. The equivalence is obtained in the natural way. Each point in the deformation space determines a marked quasifuchsian subgroup $(\sigma, \Gamma)$, which in turn determines a marked Riemann surface $(D_1/\Gamma, \sigma)$ in $T_p$. Theorem 1 is proved.

5. Intrinsic global coordinates. To use Theorem 1 we must choose an automorphism $\theta$. Let $(\sigma, \Gamma)$ be a marked quasifuchsian group with $\sigma$ given by (1). Then $\Gamma$ is the free group on $A_1, \ldots, B_p$, modulo the relation (2). Put

$$K_0 = I, \quad K_j = \prod_{i=1}^{j} C_i, \quad 1 < j < p,$$

and define $\theta$ on generators by

$$\theta(A_j) = K_{j-1}A_jK_{j-1}^{-1}, \quad \theta(B_j) = K_jB_jK_{j-1}^{-1}, \quad 1 < j < p.$$

It is easy to prove by induction on $j$ that

$$\theta(K_j) = K_j^{-1}, \quad 1 < j < p.$$

The case $j = p$ of (7) shows that $\theta$ preserves the relation (2) and does indeed define an automorphism of $\Gamma$. It is clear from (6) and (7) that $\theta$ has order two.
Every automorphism of $\Gamma$ is induced by some diffeomorphism of $D_1/\Gamma$, and any diffeomorphism that induces $\theta$ is sense-reversing since on the level of homology $\theta$ fixes each $B_j$ and reverses each $A_j$. We can therefore apply Theorem 1 to $\theta$.

**Theorem 2.** If $\theta$ is defined by (6), the normalized group $(\sigma, \Gamma)$ given by Theorem 1 is determined by $B_1, \ldots, B_p$. The multipliers of the $B_j$, the repulsive fixed points of $B_2, \ldots, B_p$, and the attractive fixed points of $B_3, \ldots, B_p$ are a global coordinate system for $T_p$.

**Proof.** Let $(\sigma, \Gamma)$ be given by Theorem 1. Since $\theta$ has order two, there is a Möbius transformation $F$ such that

$$F^2 = I \text{ and } \theta(\gamma) = F\gamma F^{-1} \text{ for all } \gamma \in \Gamma.$$  

Formula (6) implies by induction on $j$ that

$$K_j = \theta(B_1 \cdots B_j)(B_j \cdots B_1)^{-1}$$

$$= F(B_j \cdots B_1)F^{-1}(B_j \cdots B_1)^{-1}, \quad 1 < j < p.$$  

Taking $j = p$ in (9) we see that $F$ is the unique Möbius transformation of order two that commutes with the loxodromic transformation $B_p \cdots B_1$. Thus the $B_j$ determine $F$ and, by (9), each $K_j$.

Now put

$$F_j = FK_{j-1}A_j, \quad 1 < j < p.$$  

We claim that $F_j$ is the unique Möbius transformation of order two that commutes with $B_j$. First, $F_j$ is not the identity map since it interchanges the regions $D_1$ and $D_2$. Next,

$$F_j^2 = FK_{j-1}A_jFK_{j-1}A_j = \theta(K_{j-1}A_j)K_{j-1}A_j = I,$$

by (6), (7), and (8). Finally,

$$F_jB_jF_j^{-1} = F_jB_jF_j = \theta(K_{j-1}A_jB_j)K_{j-1}A_j$$

$$= A_j^{-1}K_{j-1}K_jB_jA_j = A_j^{-1}C_jB_jA_j = B_j,$$

which proves our claim.

Since the $B_j$ uniquely determine $F$ and the $F_j$, formulas (9) and (10) show that the $B_j$ determine $(\sigma, \Gamma)$. That proves the first statement of Theorem 2. The second statement follows easily. Indeed, the fixed points and multipliers of the $B_j$ are holomorphic functions on the deformation space of the Kleinian group generated by $\Gamma$ and $F$ (see §8 of [3]), hence on $T_p$. Since these fixed points and multipliers determine the $B_j$, and hence $(\sigma, \Gamma)$, we have defined an injective holomorphic map from $T_p$ into $C^*$. The theorem is proved.

We remark in conclusion that $\{B_1, \ldots, B_p\}$ generates a Schottky group of genus $p$, and our coordinates on $T_p$ give an injective holomorphic map of $T_p$ into the space of Schottky groups of genus $p$. We will study the geometry of that map in a forthcoming paper.
REFERENCES


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